IN-PLANE STABILITY OF UNIFORMLY PRESTRESSED CIRCULAR ARCHES

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1. SUMMARY

The stability of arches has been studied abundantly in the past and many analytical solutions for high and shallow arches have been developed. Numerical solutions for in-plane and out-of-plane stability under different loadings have also been published. However, in these studies the arches are assumed to be stress-free in their initial, unloaded condition. The case of prestressed arches has not been addressed so far. In this paper the influence of prestressing in the in-plane stability of arches is investigated. Prestressing may be unintentional, due to the fabrication process of the arch, or intentional, if it proves to be beneficial for the arch's structural behavior. The simpler case of elastic prestressing is addressed here, while the more complex case of elasto-plastic residual stresses due to the fabrication process will be the objective of future studies. Variational methods are used to obtain analytical solutions for the buckling loads of simply-supported, uniformly prestressed circular arches under uniform radial pressure. The solutions can be easily extended to other cases of loads and boundary conditions.

Keywords: Arches, stability, prestress, cold roll bending, elastic buckling.

2. STATE OF ART ON IN-PLANE STABILITY OF ARCHES

When subjected to compressive forces, circular arches with restrained lateral displacements may buckle in their plane in an asymmetrical mode, or in a symmetrical mode, which is then called snap-through. From the viewpoint of buckling analysis, it is useful to distinguish two basic types of arches: high arches and shallow arches. For high arches, the centre line of the arch may be considered as incompressible, whereas for flat arches, its shortening is important [1]. What is often referred to as the classical theory of buckling, is actually based on inextensibility of the arch and provides thus analytical solutions for high arches. In this approach, the elastic buckling load is determined by introducing second order effects of the compressive stress in the equilibrium equation of the bending moment, like in elementary theory of column buckling (see for example [2]).

For shallow arches, some authors propose to use a sinusoidal curve approximation of the circular arch and a development of the external load in Fourier series [1,3]. This solution has provided interesting results, especially for imperfection sensitivity. A significant improvement in the study of shallow arches was Schreyer's and Masur's exact solution of the nonlinear equilibrium equations subjected to either uniform pressure or concentrated central load [4]. This solution, based on the energy method, was adopted by many authors, among them the group of Trahair and Bradford from the University of New South Wales in Sidney [5]. Their research aims at understanding the influence of the non-linear pre-buckling deformation on the buckling load of arches and it confirmed that they can be only neglected for high arches. They applied their method to a wide variety of problems, from elastic stability of steel sections [6] to creep buckling of concrete arches [7] and also flexural torsional buckling of arches under compression [8] or bending [9].

Few authors have been concerned by the influence of prestress on the stability of circular arches. Clifton used the analytical solution of Schreyer and Masur, in which he introduced an initial normal prestress and quantified its influence on buckling loads and buckling modes [10]. Except of this paper, most research on the subject of prestressed arches focused on the *elastica*, which is actually the post-buckled curve of a straight beam. Thompson and Hunt developed an energy model based on Taylor's series, applied it to a trigonometric first order approximation of the *elastica* and confirmed Clifton's results. They found that the buckling load was linearly dependant on the average axial prestress, and that the higher the initial compression, the smaller the critical buckling load [3].

A lot of numerical work (and also experimental work in a centrifuge [11]) was also done in the University of Maryland, where Mirmiran, Amde and Wolde-Tinsae developed an innovative concept of dome made of prestressed sandwich arches [12]. They studied first the simple *elastica* and found that "residual compressive stresses that result when the prestressed arch framework is formed have a negative effect on the stability of its member" [13]. Then, improving their concept by transforming it into a sandwich framework, they concluded that if the "prestressing is applied separately to the layers of the sandwich section, unlike single elastica arches, the resulting arch will be as stable as its rigid or non-prestressed counterpart" [12]. This surprising result is actually linked with the fact that after gluing together the different layers in the sandwich, the inertia of the arch is dramatically increased, so that the sum of the prestress in the layers is negligible in comparison to the buckling load of the arch.

Hence, the stability of circular arches has been abundantly studied in the past, analytically and numerically. In most studies, the arches are assumed to be stress-free in their initial, unloaded condition. Indeed, with the exception of the very specific case of *elastica*, the case of prestressed arches has not been studied so far. This paper will thus study the influence of uniform prestress and be presented as follows. The next section describes the energy method used for the analytical calculation of buckling load for high arches without prestress. Section 4 is then concerned with the introduction in the model of uniform prestress in the arch. A brief discussion concludes the paper and emphasizes the future development of the model for the design of steel arches taking into account the residual stresses induced by the forming process.

3. FIRST ORDER MODEL FOR ELASTIC BUCKLING OF CIRCULAR ARCHES

For simplicity, the article focuses on circular arches subjected to uniform radial pressure (figure 1). Other loading cases such as concentrated forces or vertical uniform load will be discussed in future work; it has been established, however, that they do not really influence the main results of this article. Pinned boundary conditions have been chosen because they are most commonly used in practice. We shall also show in a future paper that this choice does not qualitatively alter our main results. For the determination of the local equation of the behavior and of the buckling load of the arch, calculations will be based on an energy approach. It will be assumed that the beam is sufficiently slender so that Euler-Bernoulli model describes with good accuracy the strains during the whole transformation, which is considered as perfectly elastic and as inducing only small displacements and small rotations.

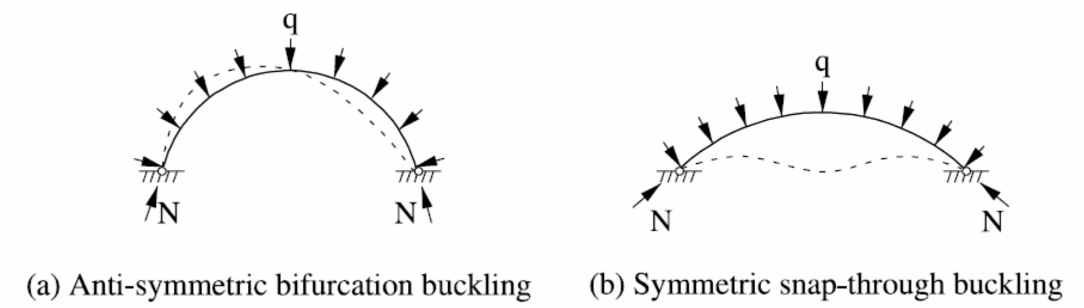

Figure 1: Buckling modes of circular arches under uniform radial pressure.

3.1 Principle of the variational method

In the arch of figure 2, R is the radius and θ the characteristic angle defining a point located on the arch. Maximum values for θ are $\pm\alpha$. u and v are the tangential and radial displacements, respectively, positive when directed toward the center of the circle.

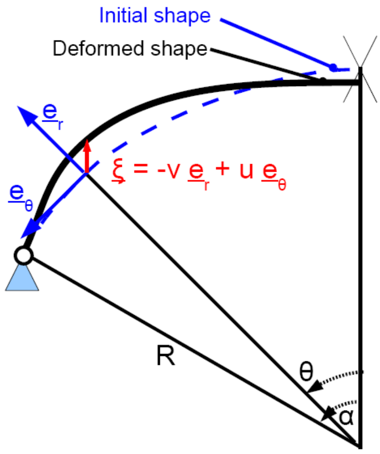

Figure 2: Local axes of the arch.

According to our hypothesis, the axial strain ε and curvature κ in the beam are given by:

$$\varepsilon = \frac{u'}{R} - \frac{v}{R} + \frac{1}{2}\left(\frac{u}{R} + \frac{v'}{R}\right)^2, \qquad \kappa = \frac{1}{R}\left(\frac{u'}{R} + \frac{v''}{R}\right) \tag{1}$$

It must be noted here that this non-linear expression slightly differs from that of Pi and Bradford [5] who are mainly concerned with the behavior of shallow arches. Indeed, with small values of α ($<\pi/4$ or $<\pi/3$ depending of the inertia properties of the section), one can show with a standard elastic structural analysis that the ratio $|u/v'|$ is very small so that u can be neglected in the term in brackets in the expression of axial strain. This assumption allows Pi and Bradford to take initial strain into account for the determination of critical loads [5]. We, here, intend only to calculate the elastic buckling load with first order analysis and therefore can keep a more general expression for ε.

For conciseness of expressions, we introduce non-dimensional displacements $\overline{u}$ and $\overline{v}$, which denote the ratios *u/R* and *v/R*, respectively. Expressions in (1) are now written as following:

$$\varepsilon = \overline{u}' - \overline{v} + \frac{1}{2}\left(\overline{u} + \overline{v}'\right)^2, \qquad \kappa = \frac{\overline{u}' + \overline{v}''}{R} \tag{2}$$

The strain energy W^{def} associated with these deformations can be calculated from the Young's modulus *E*, the section area *S* and the inertia *I* of the beam, while the potential energy W^{ext} of the external forces can also be evaluated from the radial pressure p and the radial displacements:

$$W^{def} = \frac{1}{2}\int_{-\alpha}^{\alpha} EI\kappa^2 Rd\theta + \frac{1}{2}\int_{-\alpha}^{\alpha} ES\varepsilon^2 Rd\theta,\ W^{ext} = -\int_{-\alpha}^{\alpha} pR^2\overline{v}d\theta \tag{3}$$

At equilibrium, the total energy $W^{tot} = W^{ext} + W^{def}$ is minimum. Therefore, for any small displacements $\delta\overline{u}$ and $\delta\overline{v}$ around the equilibrium solution, the variation of the total energy must be equal to zero:

$$\forall\delta\overline{u} \text{ and } \forall\delta\overline{v} \quad \delta W^{def} + \delta W^{ext} = 0 \tag{4}$$

In the above equation, one has to expand the different terms and to explicit its dependency in $\delta\overline{u}$ and $\delta\overline{v}$. From equations (3) the variation of energy of external forces and strain energy are:

$$\delta W^{ext} = -\int_{-\alpha}^{\alpha} pR^2\delta\overline{v}d\theta, \qquad \delta W^{def} = \int_{-\alpha}^{\alpha} EI\kappa\delta\kappa Rd\theta + \int_{-\alpha}^{\alpha} ES\varepsilon\delta\varepsilon Rd\theta \tag{5}$$

In this expression, *δε* and *δκ* denote the variation of axial strain and curvature, evaluated from (2):

$$\varepsilon = \delta\overline{u}' - \delta\overline{v} + \left(\overline{u} + \overline{v}'\right)\left(\delta\overline{u} + \delta\overline{v}'\right),\ \delta\kappa = \frac{\delta\overline{u}' + \delta\overline{v}''}{R} \tag{6}$$

Introducing (6) into (5), integrating by parts and accounting for the boundary conditions, one obtains:

$$\int_{-\alpha}^{\alpha}\left(-EI\kappa' - ES\varepsilon' + ESR\varepsilon\left(\overline{u} + \overline{v}'\right)\right)\delta\overline{u}Rd\theta + \int_{-\alpha}^{\alpha}\left(EI\kappa'' - ES\varepsilon - ESR\left(\varepsilon\left(\overline{u} + \overline{v}'\right)\right)' - pR^2\right)\delta\overline{v}Rd\theta = 0 \tag{7}$$

This expression must be equal to zero for any displacements $\delta\overline{u}$ and $\delta\overline{v}$, so that both terms in brackets in the integral must be zero:

$$-EI\kappa' - ES\varepsilon' + ESR\varepsilon\left(\overline{u} + \overline{v}'\right) = 0, \qquad EI\kappa'' - ES\varepsilon - ESR\left(\varepsilon\left(\overline{u} + \overline{v}'\right)\right)' - pR^2 = 0 \tag{8}$$

These are the two equations of local equilibrium. To simplify them we introduce two new notations, the slenderness of the arch λ=i/R, where i is the inertia radius of the cross-section, and a nondimensional expression of the load, $\overline{\mathrm{p}} = \mathrm{pR}^3/\mathrm{EI}$. By differentiating the first of equations (8) and adding it to the second, we get the final equations for non-linear local equilibrium of an arch under uniform pressure:

$$\varepsilon'' + \varepsilon = -\lambda\overline{p}, \qquad -\lambda R\kappa' - \varepsilon' + \varepsilon\left(\overline{u} + \overline{v}'\right) = 0 \tag{9}$$

3.2 Evaluation of buckling load

The non-linearity introduced by the last term in the second of equations (9) does not allow us to find an exact analytical solution. Nevertheless, the study of the linear problem associated with these equations is straightforward and leads to:

$$\varepsilon^{lin} = -\lambda\overline{p}\left(1 - \lambda\frac{\sin\alpha}{D(\alpha)}\cos\theta\right), \qquad \kappa^{lin} = \frac{\lambda\overline{p}}{R}2\sin\alpha\left(\cos\alpha - \cos\theta\right) \tag{10}$$

where *D(α)* is a trigonometric function of *α* inferior to *α*. We remark that when *λ* is small (which is generally the case for arches used for structural purposes) the axial strain or the compression in the arch can be assumed constant. Therefore, we will suppose that also during buckling the compression in the arch will remain constant and equal to *N= -pR*. Equations (9) become thus:

$$\varepsilon = -\lambda \bar{p}, \qquad R\kappa' + \bar{p}\left(\bar{u} + \bar{v}'\right) = 0 \tag{11}$$

To solve equation (11), we have to go back to the expression of κ given by (2) and introduce a new variable $\gamma=(u+v')/R$, which actually represents the local rotation of the beam. Equation (11) becomes:

$$\gamma'' + \bar{p}\gamma = 0 \tag{12}$$

The solution of this equation is then:

$$\gamma = A\sin\sqrt{\bar{p}}\theta + B\cos\sqrt{\bar{p}}\theta \tag{13}$$

A and *B* are two constants which will be determined by the boundary conditions at ±α where the bending moment is zero (i.e. $\gamma'(\pm\alpha) = 0$):

$$A\cos\sqrt{\bar{p}}\alpha = B\sin\sqrt{\bar{p}}\alpha = 0 \tag{14}$$

When the solution is not equal to zero, it is unbounded, so that we may consider that the arch buckles. Thus, there is a non-zero solution if:

i) $\sqrt{\bar{p}}\alpha = n\pi$ and $B \neq 0$, corresponding to antisymmetrical buckling modes,

ii) $\sqrt{\bar{p}}\alpha = \left(n + 1/2\right)\pi$ and $A \neq 0$, corresponding to symmetrical buckling modes.

Considering the hypothesis that the compression remains constant during buckling, we must point out that the value $n=0$ is impossible for symmetrical buckling so that the first critical values are then the values currently given by most authors for linear elastic buckling of circular arches:

$$N_{cr}^{ant} = \frac{\pi^2 EI}{\left(R\alpha\right)^2}, \qquad N_{cr}^{sym} = \frac{9}{4}\frac{\pi^2 EI}{\left(R\alpha\right)^2} \tag{15}$$

4. EQUATIONS OF CIRCULAR ARCHES UNDER UNIFORM PRESTRESS

We suppose now that for some reason, intentional or not, a prestress is introduced in the arch before it is loaded. We assume that this prestress is uniform along the arch, which means that it is independent of θ. The prestressing strain can be divided into two parts, one which is constant in the section and will be denoted by ε_0, and another, denoted κ_0, which is a function of the distance *y* to the neutral axis of the beam, but has zero average. If the variable part of the prestress, κ_0, creates a bending moment, then this bending moment is uniform along the arch and can equilibrated by introducing additional external moments at both ends. The problem studied in this section is illustrated in figure 3.

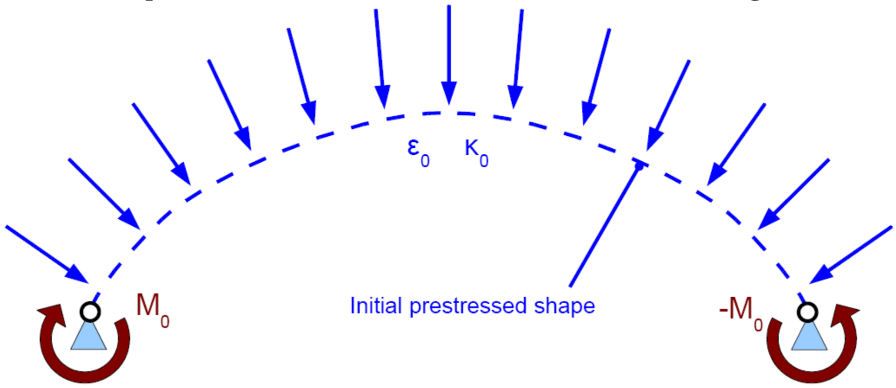

Figure 3: Prestressed arch with additional bending moments

4.1 Assessment of energies induced by the prestress

The prestress induces a permanent strain energy stored into the beam:

$$W^0 = \int_{-\alpha}^{\alpha} ES\varepsilon_0{}^2 Rd\theta + \iint_S \int_{-\alpha}^{\alpha} E\kappa_0(y)^2\, Rd\theta dydz \tag{16}$$

Then, when the arch is loaded and deforms under pressure, another energy W^{pres} must be introduced. It is linked with the prestress and corresponds to the work of internal forces due to the prestress with the displacements of the beam:

$$W^{pres} = \int_{-\alpha}^{\alpha} ES\varepsilon_0\, \varepsilon Rd\theta + \iint_S \int_{-\alpha}^{\alpha} E\kappa_0(y)\, y\kappa Rd\theta dydz \tag{17}$$

Thus, the total energy of the loaded arch is $W^{tot} = W^0 + W^{pres} + W^{ext} + W^{def}$. Next, like previously, the local equations of equilibrium are determined by demanding that the variation of the total energy is equal to zero for any small displacements $\delta\overline{u}$ and $\delta\overline{v}$ around the solution. Noting that δW^{def} is unchanged, one has just to evaluate δW^0, δW^{pres} and δW^{ext}:

$$\delta W^0 = 0 \tag{18}$$

$$\delta W^{pres} = \int_{-\alpha}^{\alpha} ES\varepsilon_0\, \delta\varepsilon Rd\theta + \iint_S \int_{-\alpha}^{\alpha} E\kappa_0(y)\, y\delta\kappa Rd\theta dydz \tag{19}$$

$$\delta W^{ext} = -\int_{-\alpha}^{\alpha} pR^2\delta\overline{v}d\theta - M_0\delta\gamma(\alpha) + M_0\delta\gamma(-\alpha) \tag{20}$$

Then, δW^{pres} is integrated by parts, taking into account the boundary conditions:

$$\delta W^{pres} = \int_{-\alpha}^{\alpha} ES\varepsilon_0\left(\gamma\delta\overline{u} - \delta\overline{v}\right)Rd\theta + M_0\delta\gamma(\alpha) - M_0\delta\gamma(-\alpha) \tag{21}$$

4.2 Evaluation of the buckling load

To evaluate the buckling load we sum the above expression with expressions (5) and require that the terms in the integral must be equal to zero for any small displacements $\delta\overline{u}$ and $\delta\overline{v}$. We obtain then the equilibrium equations for the prestressed arch:

$$-\lambda\gamma'' - \varepsilon' + (\varepsilon + \varepsilon_0)\gamma = 0, \qquad \lambda\gamma''' - \varepsilon - \varepsilon_0 - (\varepsilon\gamma)' - \lambda\overline{p} = 0 \tag{22}$$

Once again, we combine equations (22) to obtain a simpler equation for ε:

$$\varepsilon + \varepsilon'' = -\lambda\overline{p} - \varepsilon_0 \tag{23}$$

Then we assume that the compression remains constant during buckling, so that we neglect the second derivative of ε in (23). Equation (22) changes into:

$$\varepsilon = -\lambda\overline{p} - \varepsilon_0, \qquad \gamma'' + \overline{p}\gamma = 0 \tag{24}$$

Considering these equations together with (11) and (12), we remark that a uniform "pure bending" prestress has no influence on the equilibrium equations and that only ε_0, the prestress which is constant in the section, will have consequence on the behavior and the buckling load. Indeed, following the same argument as for the arch without prestress, we can deduce the values of the critical external pressure which will cause the first symmetrical and antisymetrical buckling modes:

$$(pR)_{cr}^{ant} = \frac{\pi^2 EI}{(R\alpha)^2} - ES\varepsilon_0, \qquad (pR)_{cr}^{sym} = \frac{9}{4}\frac{\pi^2 EI}{(R\alpha)^2} - ES\varepsilon_0 \tag{25}$$

The critical pressure is thus linearly dependant on the axial prestress. As expected, if the prestress is a compression ($\varepsilon_0 > 0$), the critical buckling loads of the prestressed arch will be smaller than those of the arch without prestress. This result is similar to that found by Thompson and Hunt for shallow *elasticae* [3] or Clifton for shallow circular arches [10].

5. SUMMARY, CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

A variational approach has been used to obtain analytical solutions for the buckling loads of simply-supported, uniformly prestressed circular arches under uniform radial pressure. It has been deduced that the bending part of prestressing has no influence on the buckling loads, while the critical pressure is linearly dependant on the axial part of prestress. The solutions can be easily extended to other cases of loads and boundary conditions, which have not been presented here due to lack of space.

Future research work of the authors will investigate this problem in the elasto-plastic range of material behaviour, both during prestressing, and under service loads. Of particular interest is the case of plastic stresses and strains induced to the arch during the process of cold bending, which is routinely employed for the curving of steel members at ambient temperature, and involves substantial deformations of the element. During bending, yielding takes place continuously along the length, uniformly altering the residual stress distribution. Assuming that the whole process is perfectly continuous and that the final radius of curvature is constant along the arch, that the steel is perfectly plastic and that the whole section yields during bending, it is possible to analytically evaluate residual stresses induced by roll bending. Besides ensuring that these stresses will not induce local buckling during bending, it will then also be possible to introduce them as initial prestress and investigate the subsequent behaviour and stability of the arch under service loads.

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ΕΥΣΤΑΘΕΙΑ ΕΝΤΟΣ ΕΠΙΠΕΔΟΥ ΟΜΟΙΟΜΟΡΦΑ ΠΡΟΕΝΤΕΤΑΜΕΝΩΝ ΚΥΚΛΙΚΩΝ ΤΟΞΩΝ

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ΠΕΡΙΛΗΨΗ

Η ευστάθεια τόξων έχει μελετηθεί εκτενώς στο παρελθόν και έχουν αναπτυχθεί πολλές αναλυτικές λύσεις για υψηλά και χαμηλά τόξα. Επίσης έχουν δημοσιευθεί αριθμητικές λύσεις για τη ευστάθεια τόξων υπό διάφορα φορτία, εντός και εκτός επιπέδου. Όμως σε αυτές τις μελέτες τα τόξα θεωρήθηκαν ελεύθερα τάσεων στην αρχική, αφόρτιστη κατάστασή τους. Η περίπτωση προεντεταμένων τόξων δεν έχει για την ώρα αντιμετωπιστεί. Στο παρόν άρθρο εξετάζεται η επιρροή της προέντασης στην εντός επιπέδου ευστάθεια τόξων. Η προένταση μπορεί να είναι αθέλητη, π.χ. λόγω του τρόπου κατασκευής του τόξου, ή ηθελημένη, εάν αποδεικνύεται ότι μπορεί να είναι ευεργετική για την μετέπειτα συμπεριφορά του τόξου υπό φορτία. Εδώ εξετάζεται η απλούστερη περίπτωση ελαστικής προέντασης, ενώ η πιο σύνθετη ελαστο-πλαστική περίπτωση, που συνδέεται με παραμένουσες τάσεις λόγω της διαδικασίας παραγωγής θα αντιμετωπιστεί στο μέλλον. Χρησιμοποιείται η μέθοδος λογισμού των μεταβολών για την εύρεση αναλυτικών λύσεων για τα φορτία λυγισμού αμφιέρειστων, ομοιόμορφα προεντεταμένων κυκλικών τόξων υπό ομοιόμορφη ακτινική πίεση. Οι λύσεις μπορούν εύκολα να επεκταθούν και για άλλες περιπτώσεις φορτίων και συνοριακών συνθηκών.