A ROD ELEMENT FORMULATION FOR THE NONLINEAR DYNAMIC ANALYSIS OF TRUSSES.

Savvas P. Triantafyllou, Vlasis K. Koumousis
Institute of Structural Analysis & Aseismic Research
National Technical University of Athens
NTUA, Zographou Campus GR-15773, Athens, Greece
e-mail: savtri@central.ntua.gr, vkoum@central.ntua.gr

ABSTRACT

In this work, an alternative rod element formulation is proposed for the nonlinear dynamic analysis of trusses. The classical, geometrically nonlinear elastic rod element formulation is extended by implicitly defining new, hysteretic, degrees of freedom, subjected to an evolution equation of the Bouc Wen type with kinematic hardening. An interpolation field is proposed for the new degrees of freedom, which are regarded as hysteretic strains. By means of the principle of virtual work a geometrically nonlinear elastoplastic stiffness matrix is derived. This stiffness matrix together with the hysteretic evolution equations fully describes the constitutive behavior of the element. Solutions are obtained by simultaneously solving the three sets of governing equations of the structure, namely the global equilibrium equations, global compatibility equations and local constitutive equations. A Livermore solver for stiff differential equations is implemented. Following this approach, the linearization of the constitutive relations is avoided, contrary to the usual step – by – step solution approaches. Furthermore, stability problems can be studied as a dynamic phenomenon. The efficiency of the proposed method is demonstrated with a characteristic example.

1 Introduction

With the advancement in computer technology, many researchers direct their efforts toward nonlinear analysis of structures. In this field, two major approaches have been adopted. The displacement based finite element approach and the force based finite element approach. The primary unknowns in the latter method are the internal element forces instead of the nodal displacements used in the former method. Amjad et al. (2001) and Barham et al. (2000, 2005) made use of the large increment method in order to solve the nonlinear equations of motion.

Nonlinearities in a structural system can have a profound effect on its transient structural response. Trusses usually have higher natural frequencies compared to relevant solid structures, because of their high stiffness-to-mass ratio. The nonlinearity of trusses under dynamic loading can stem from various origins: (i) geometrical-due to the variations in the geometrical properties of the structure as the load progresses; (ii) material-due to the inherent nonlinear behaviour of the materials under load; (iii) inertia-depending on the dynamic motion and the structural deformations; and (iv) damping depending on the structural joints and material. In this work a novel analysis procedure together with a fully nonlinear rod element are presented. The rod element is constructed on the grounds of an updated Lagrangian formulation together with a Bouc Wen inelastic law. Equilibrium and compatibility equations are expressed for the whole structure in terms of global nodal forces and global nodal displacements. In this way, equilibrium and compatibility equations are linear. Material nonlinearity is maintained at the elemental level through proper implementation of the Bouc-Wen hysteretic rule.

2 The Bouc-Wen hysteretic rule

The smooth model presented herein is a variation of the model originally proposed by Bouc (1967) and modified by several others (Wen 1976, Baber Noori 1985, Cherng 1991, Reinhorn 2000, 2003). The model is developed in the context of uniaxial stress-strain relationships. The use of such a hysteretic constitutive law is necessary for the effective simulation of the behavior of structures under cyclic loading, since often structures that undergo inelastic deformations and cyclic behavior weaken and loose some of their stiffness and strength. The model can be visualized as a parallel combination of a linear and a nonlinear element, as shown in Figure 1. The generalized stress - strain relation is given by:

$$\sigma(t) = \sigma_y \left[\alpha \frac{\varepsilon(t)}{\varepsilon_y} + (1-\alpha)\frac{z(t)}{\varepsilon_y} \right] = \alpha E \varepsilon(t) + (1-\alpha) E z(t) \qquad (1)$$

where σ_y is the yield stress; ε_y is the yield strain; E is the initial Young Modulus, α is the ratio of the post-yield to the initial elastic stiffness and z(t) is the hysteretic component defined in relation (2) below.

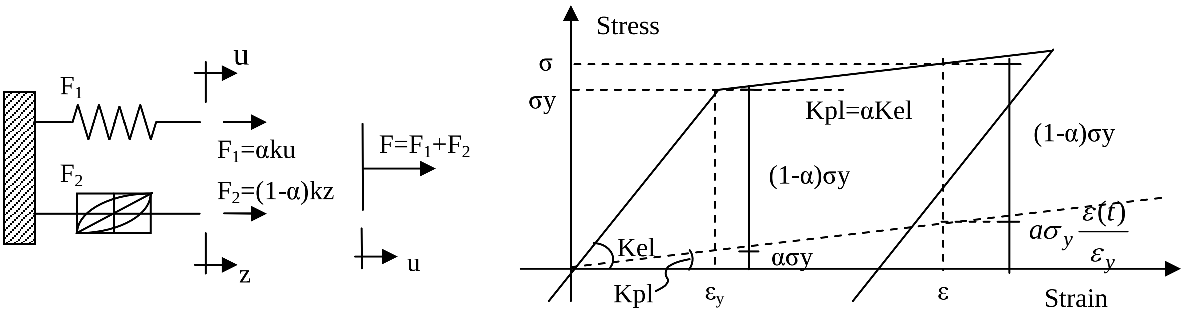


Figure 1. Bouc-Wen Hysteretic Model

The hysteretic function z(t) is obtained from the non-linear differential equation:

$$\dot{z}(t) = f(\dot{\phi}(t), z(t)) = \dot{\phi}\left[1 - \left|\frac{z}{\phi_y}\right|^n \left(\beta + \gamma \operatorname{sgn}\left(z\dot{\phi}\right)\right)\right] \qquad (2)$$

where dot designates differentiation with respect to time. It can be easily noticed that the hysteretic curvature should comply with the following rules:

$$z = e, \text{ in the elastic region}$$
$$z = e_y, \text{ in the inelastic region}$$

3 Space Truss Element Formulation

3.1 Kinematic Relations

In the framework of large displacements, the axial strain of the rod element is expressed as:

$$\varepsilon_x = e_x + \eta_x \tag{3}$$

where e_x is the linear part of the strain:

$$e_x = \frac{\partial u_x}{\partial x} \tag{4}$$

and η_x is the nonlinear part of the axial strain:

$$\eta_x = \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] \tag{5}$$

3.2 Hysteretic Strain

The following nonlinear law applies for the stress-strain relation:

$$\sigma_x(s,t) = \alpha E \varepsilon_x(s,t) + (1-\alpha) E z(s,t) \tag{6}$$

where the axial strain ε_x is derived from equations (3) to (5) and z can be considered to be the hysteretic part of the strain subjected to the evolution equation described in relation (2).

3.3 Interpolation Field and Functions

Implementing the linear interpolation functions for the displacement field one gets:

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{bmatrix} 1-\frac{s}{L} & 0 & 0 & \frac{s}{L} & 0 & 0 \\ 0 & 1-\frac{s}{L} & 0 & 0 & \frac{s}{L} & 0 \\ 0 & 0 & 1-\frac{s}{L} & 0 & 0 & \frac{s}{L} \end{bmatrix} \{d\} \tag{7}$$

$$\{d\} = \{u_x^1 \quad u_y^1 \quad u_z^1 \quad u_x^2 \quad u_y^2 \quad u_z^2\}^T$$

The strain field is derived from the displacement field, by substituting (7) in (4), (5) thus leading to the following formulation:

$$\varepsilon = ([B]_L + [B]_{NL})\{d\} \tag{8}$$

where:

$$[B]_L = \begin{bmatrix} \frac{1}{L} & 0 & 0 & \frac{1}{L} & 0 & 0 \end{bmatrix}$$

$$[B]_{NL} = \begin{bmatrix} \frac{\Delta u_x}{2L^2} & -\frac{\Delta u_y}{2L^2} & -\frac{\Delta u_z}{2L^2} & \frac{\Delta u_x}{2L^2} & \frac{\Delta u_y}{2L^2} & \frac{\Delta u_z}{2L^2} \end{bmatrix} \quad (9)$$

$$\Delta u_x = -u_x^1 + u_x^2, \Delta u_y = -u_y^1 + u_y^2, \Delta u_z = -u_z^1 + u_z^2$$

An interpolation scheme is also introduced for the hysteretic part of the strain. Assuming a constant strain distribution, the hysteretic part of the strain is obtained as:

$$\{z\} = [B]_z \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = [-1 \quad 1] \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \quad (10)$$

where z_1 and z_2 are nodal hysteretic strains subject to the evolution law of relation (2).

3.4 Stiffness Matrix

The principle of virtual work is stated by the following relation:

$$\int_V \delta\varepsilon \cdot \sigma \, dV = \{d\}^T \{P\} \quad (11)$$

By substituting (6) and (3) in (11) one gets:

$$\int_V (\delta e_x + \delta\eta_x)\left[\alpha E(e_x + \eta_x) + (1-\alpha)Ez\right] dV = \{d\}^T \{P\} \quad (12)$$

Taking into consideration equations (7) to (10) and after the necessary algebraic manipulations the following constitutive relation is derived:

$$\underbrace{\left[\overset{(6x6)}{k_e} + \overset{(6x6)}{k_g} + \overset{(6x6)}{s_1} + \overset{(6x6)}{s_2} + \overset{(6x6)}{s_3} \;\middle|\; \overset{(6x2)}{k_z} \right]}_{(6x8)} \underset{(8x1)}{\{d \mid z\}} = \underset{(6x1)}{\{P\}} \quad (13)$$

$$\{P\}^T = \{P_x^1 \quad P_y^1 \quad P_z^1 \quad P_x^2 \quad P_y^2 \quad P_z^2\}$$

Matrices k_g, k_s, s_1, s_2, s_3 are the same as in the updated Lagrangian formulation of the two node truss element [10] multiplied by α and K_z is defined as:

$$[K_z] = (1-\alpha)EA \begin{bmatrix} 1+\frac{\Delta u_x}{2L} & \frac{\Delta u_y}{2L} & \frac{\Delta u_z}{2L} & -1-\frac{\Delta u_x}{2L} & -\frac{\Delta u_y}{2L} & -\frac{\Delta u_z}{2L} \\ 1-\frac{\Delta u_x}{2L} & -\frac{\Delta u_y}{2L} & -\frac{\Delta u_z}{2L} & 1+\frac{\Delta u_x}{2L} & \frac{\Delta u_y}{2L} & \frac{\Delta u_z}{2L} \end{bmatrix}^T \quad (14)$$

4 Solution procedure

4.1 Formulation of the governing equations

The main advantage of the proposed method is that it separates the problem into two sets of equations and is based on the node method as described in [9]. The first set consists of the global linear equilibrium and compatibility equations, while the second one of local nonlinear constitutive equations, together with the hysteretic evolutionary equations. These sets of equations are solved simultaneously using a Runge–Kutta 4-5th order integrator. The primary unknowns of the method are the generalized forces of the elements ($3n_{el}$), the nodal displacements expressed in the global coordinate system (3nodes), and the hysteretic strains ($2n_{el}$). The dynamic equations of equilibrium are written in the following mixed form:

$$[M]\{\ddot{U}\} + [C]\{F\} = \{P(t)\} \quad (15)$$

where {U} is the vector of nodal displacements in the global coordinate system, [C] is the equilibrium matrix of the structure, {F} is the vector of elemental forces in the local coordinated system and P(t) the vector of nodal external loads. For the case of trusses, the equilibrium matrix is merely the connectivity matrix of the structure. These equations are linear with respect to the nodal displacements and the internal forces. The compatibility equations of the problem can be formulated using the following expression:

$$[C]^T \{U\} = \{\delta\} \quad (16)$$

where {δ} is the vector of the elemental generalized displacements. Alternatively, compatibility can be accomplished by demanding:

$$U^i_{el,j} = U^i_j, \quad i = 1...nodes, \quad j = x, y, z \quad (17)$$

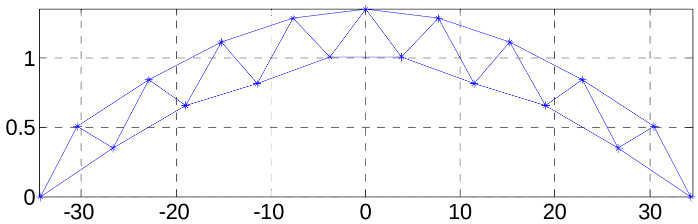
where $U^i_{el,j}$ is the elements nodal displacement expressed in the global coordinates system. The boundary conditions of the problem are also introduced in this stage. Finally, the constitutive relations are written in local level, in the form:

$$[K]_{tot} [\Lambda_z]_{tot} \{U\} = \{F\} \quad (18)$$

where $[K]_{tot}$ is a 6n x 8n block diagonal matrix, having each element's generalized stiffness matrix in its diagonal, n being the total number of elements in the structure and $[\Lambda_z]_{tot}$ is a 8nx8n block diagonal matrix, with elements the 8x8 transformation matrices of the elements consisting of the standard 6x6 transformation matrices having two additional ones in the 7th and 8th diagonal entries and zeros elsewhere, as z doesn't change in the plane being a scalar quantity

5 Numerical Example

A shallow arch is examined with a rise to span ratio of about 2%. The geometry and the mechanical properties of the truss are summarized in Figure 2. The arch is considered restrained against out of plane motions. Pinned boundary conditions are imposed at both ends of the structure. An additional mass of 3.5 KN is considered to be lumped at each node of the lower chord.



σ_y	235 MPa
α	0.02
A	$6\ 10^{-4}\ m^2$
E	210 GPa

Figure 2. Shallow Arch Configuration

Two analyses are performed. In the first, material nonlinearities are neglected while in the second both material and geometric nonlinearities are considered. A sinusoidal vertical excitation is imposed at the upper midspan node of the following form:

$$P_{ex} = 15 \sin(pt) \tag{19}$$

Four cycles of the imposed excitation are considered. In Figure 3 the time history of the vertical displacement at midspan is compared for the two analysis procedures. The difference between the two analysis procedures is substantial. Due to geometric nonlinearities the stiffness of the structure is constantly changing as depicted in the graph. If material nonlinearities are taken into account then the truss collapses at 4.1 secs.

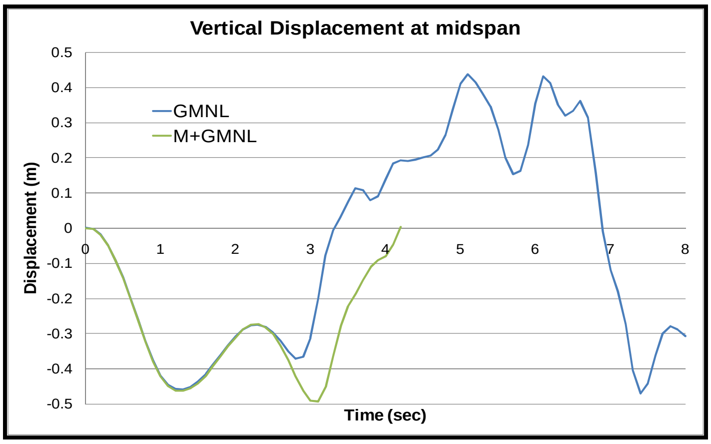


Figure 3. Vertical Displacement at midspan

Due to material yielding, the structure demonstrates hysteretic behavior as presented in the hysteretic loops of Figure 4, where the axial force of the midspan chord member is plotted against the member's axial displacement.

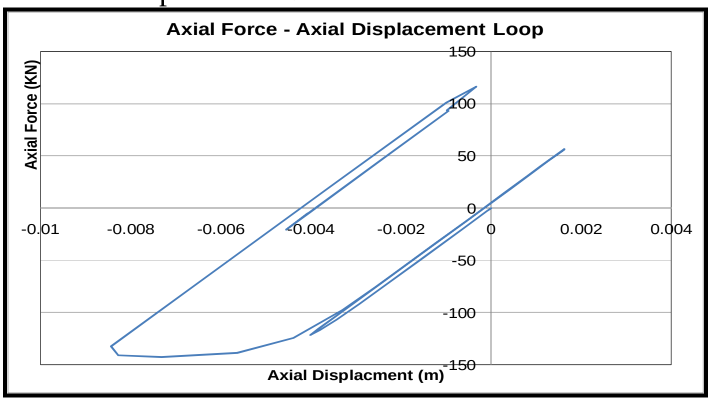


Figure 4. Nonlinear hysteresis loop due to material yield and geometric nonlinearities.

6 Conclusions

A new truss element formulation is presented, together with an alternative method for the solution of the nonlinear equations of motion. By writing down the governing equations in state space form and implementing a Livermore integrator the linearization of the constitutive equations is avoided. The Bouc-Wen hysteretic model is implemented in order to simulate the nonlinear constitutive behavior of the material, in terms of stress - strain relation. Various loops can be modeled by properly controlling the parameters of the hysteresis law, namely the “yield” strain, the smoothing parameter n, and the shape factors beta and gamma. The problem is partitioned into three sets of equations, which are solved simultaneously. The numerical examples presented demonstrate the validity of the proposed approach as well as its versatility as compared to displacement formulation.

References

[1]. Papakonstantinou, K. G., Dimizas, P. C., Koumousis, V. K., ”An Inelastic Beam Element with Hysteretic Damping”, *Shock and Vibration Journal, Special Issue ICED 2007, Vol. 15, nos. 3-4, June 2008, pp. 273-290.*

[2]. Barham W. S., Aref A. J., Dargush ,”Development of the large increment method for elastic perfectly plastic analysis of plane frame structures under monotonic load”, *International Journal of Solids and Structures, Vol. 42, July 2005, pp. 6586-6609.*

[3]. Sivaselvan M. V., Reinhorn A. M., (2003), “Nonlinear Analysis towards Collapse Simulation – A dynamic Systems Approach”, MCEER Technical Report.

[4]. Symeonov V. K., Sivaselvan M. V., Reinhorn A. M., (2000), “Nonlinear Analysis of Structural Frame Systems by the State Space Approach” , *Computer Aided Civil and Infrastructure Engineering*, Vol. 15, pp. 76-89.

[5]. Wen, Y.K., “Method of random vibration of hysteretic systems”, *Journal of Engineering Mechanics Division*, 102, 1976, pp. 249-263.

[6]. Cherng R. H. and Wen Y. K., “Stochastic Finite Element Analysis of Non-Linear Plane Trusses”, *Journal of Non-Linear Mechanics*, Vol.26, 1991, pp. 835-849.

[7]. Amjad J. A., Zaoyang G., (2001), “Framework for Finite Element Based Large Increment Method for Nonlinear Structural Problems”, *Journal of Engineering Mechanics*, Vol. 127, No. 7, pp. 739 – 746.

[8]. Erlicher Silvano, Point Nelly, (2004), “Thermodynamic admissibility of Bouc Wen type hysterisis models”, *C. R. Mecanique*, Vol. 332, pp. 51-57.

[9]. William R. Spillers, (1962), *Automated Structural Analysis: An introduction,* Pergamon Press Inc., New York.

[10]. Bathe K.J., (2007), Finite Element Procedures, Prentice Hall Engineering/Science/Mathematics, New York.

ΕΛΑΣΤΟΠΛΑΣΤΙΚΟ ΣΤΟΙΧΕΙΟ ΓΙΑ ΤΗ ΓΕΩΜΕΤΡΙΚΑ ΜΗ ΓΡΑΜΜΙΚΗ ΔΥΝΑΜΙΚΗ ΑΝΑΛΥΣΗ ΔΙΚΤΥΩΤΩΝ ΚΑΤΑΣΚΕΥΩΝ

Σάββας Π. Τριανταφύλλου, Βλάσης Κ. Κουμούσης
Εθνικό Μετσόβιο Πολυτεχνείο
Εργαστήριο Στατικής & Αντισεισμικών Ερευνών
Πολυτεχνειούπολη Ζωγράφου, GR-15773, ΑΘΗΝΑ
e-mail: savtri@yahoo.gr, vkoum@central.ntua.gr

ΠΕΡΙΛΗΨΗ

Στο άρθρο αυτό παρουσιάζεται μια εναλλακτική διατύπωση ενός μη γραμμικού στοιχείου δικτυώματος και μια νέα μέθοδος επίλυσης δικτυωτών κατασκευών. Το τυπικό στοιχείο δικτυώματος με γεωμετρικές μη γραμμικότητες κατά την προσαρμοστική διατύπωση Lagrange μετατρέπεται σε ένα ελαστοπλαστικό στοιχείο με την κατάλληλη προσθήκη υστερητικών βαθμών ελευθερίας. Ο ελαστοπλαστικός πίνακας δυσκαμψίας σε συνδυασμό με τις μη γραμμικές υστερητικές εξισώσεις επαρκούν για την πλήρη απόδοση της μη γραμμικής συμπεριφοράς του στοιχείου υπό μονοτονική ή ανακυκλιζόμενη φόρτιση. Για την επίλυση του συστήματος των μη γραμμικών εξισώσεων το πρόβλημα διατυπώνεται ως σύστημα μονοβάθμιων διαφορικών εξισώσεων το οποίο επιλύεται χρησιμοποιώντας τον αλγόριθμο Livermore. Οι εξισώσεις που διέπουν το σύστημα διακρίνονται στις γραμμικές εξισώσεις ισορροπίας της κατασκευή, τις γραμμικές εξισώσεις συμβιβαστού και τις μη γραμμικές καταστατικές εξισώσεις των μελών. Κατ’ αυτόν τον τρόπο αποφεύγεται η γραμμικοποίηση των καταστατικών εξισώσεων και αυξάνεται σημαντικά η ταχύτητα επίλυσης του προβλήματος, σε αντίθεση με τις κλασικές βήμα προς βήμα μεθόδους. Επιπλέον, τα προβλήματα ευστάθειας αντιμετωπίζονται εύκολα ως δυναμικά προβλήματα. Τέλος παρουσιάζεται ένα χαρακτηριστικό παράδειγμα το οποίο καταδεικνύει την αποδοτικότητα του αλγορίθμου επίλυσης.