A THIN WALLED BEAM ELEMENT FORMULATION FOR THE NON UNIFORM TORSION AND DISTORTION OF CLOSED SECTIONS

Emmanouil N. Chatzis, Savvas P. Triantafyllou and Vlasis K. Koumousis

Institute of Structural Analysis & Aseismic Research
National Technical University of Athens
NTUA, Zographou Campus GR-15773, Athens, Greece
e-mails: manolis72003@hotmail.com, savtri@central.ntua.gr, vkoum@central.ntua.gr

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1.SUMMARY

A beam element is proposed that takes into account the effects of non-uniform torsion and distortion of thin walled closed cross sections. Torsional warping and distortion violate the assumption that plane sections remain plane after deformation. To account for these phenomena the displacement field is extended by defining three new degrees of freedom at each end. The shape functions for the interpolation of the new degrees of freedom are derived by analytically solving the homogeneous differential equations of non uniform torsion and distortion. Consequently a (18*18) stiffness matrix is formulated, together with equivalent nodal actions for distributed loading. The effects of distortion and non uniform torsion cannot be disregarded especially for steel bridge box girder sections because they introduce significant stresses in both the longitudinal and transverse directions. Examples are presented which demonstrate the efficiency of the proposed element. Moreover, the effects of spacing of intermediate diaphragms is examined and the minimum number of diaphragms is determined that limit the additional stresses below the one tenth of max bending stresses; a threshold that Eurocode 3 specifies as the limit to use classical beam theory in bridge analysis.

2. INTRODUCTION

Bridge construction is one of the most challenging and demanding tasks of Civil Engineering. The structural systems used to span specific lengths vary from simple trusses and beams to cable stayed and suspension bridges [5]. As the result of intensive loads in vertical as well as in horizontal directions due to wind and earthquake loads the stresses and displacements developed in bridges are substantial. More sophisticated theories are needed [3],[4] capable to express additional phenomena, namely non uniform torsion and distortion [1] of the cross section, providing the appropriate relations to evaluate the stresses and warping effects. Additional normal and shearing stresses are developed that may lead to states of stress exceeding the available strength resulting into failure. Distortion is more pronounced in steel box girders sections [2] where the thickness is relatively small as compared to that of concrete sections.

Several design codes, as for example Eurocode 3, dictate that in order to use the classical theory of beams and neglect these phenomena, one must ensure that normal stresses induced by torsion and distortion should not exceed a small portion of the flexural normal stresses. In this work the minimum number of required diaphragms needed to avoid the detailed calculation of stresses and rely on the results of classical theory of beams is determined in tabular form as a function of all the nine parameters that control the problem.

3. RECTANGULAR BOX SECTIONS

In Figure 1, a typical box section of a rectangular shape is presented with all its geometrical parameters, where the points C, S and D are the Centroid, Shear center and Distortional center of the section respectively.

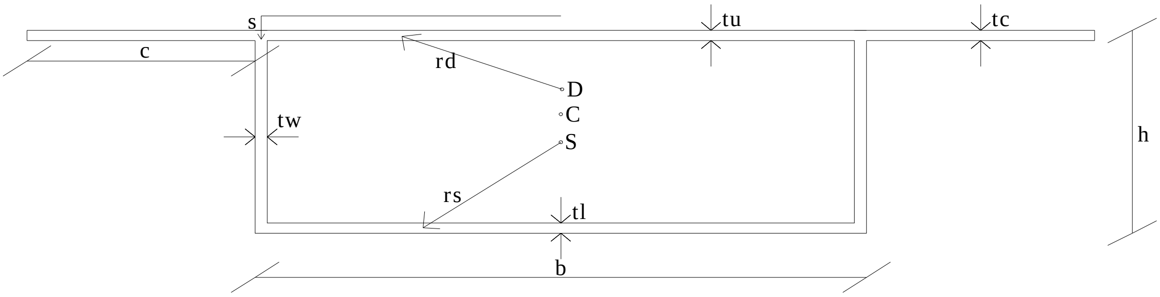


Figure 1. Typical Rectangular Box section

4. TORSIONAL WARPING

St Venant’s torsion is defined accurately for circular sections and sections prescribed in circle. Beams with a general cross section warp due to torsional loads. This means that they deform out of the plane of the cross section violating the assumption of classical beam theory according to which should remain plane.

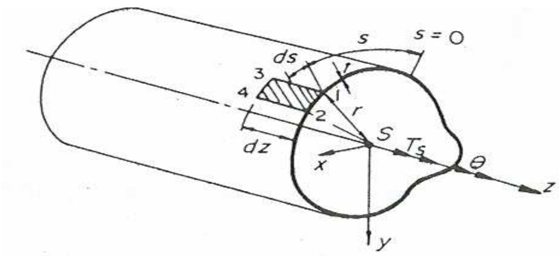


Figure 2 Generalized Section under Torsion

$$u(z,s)=\omega(s)\frac{\partial\theta(z)}{\partial z}=\omega(s)\theta' \qquad (1)$$

where ω(s) is the warping function defined as:

$$\omega=-\int_0^s r_s ds+\int_0^s \frac{\tilde{q}}{t} ds\,, \quad q_i=G\theta'\tilde{q}_i \qquad (2)$$

and r_s=the distance from the torsional center S and q is the shear flow in the cross section. When the axial displacement u is free then the beam is under pure torsion with warping, while when restrained non uniform torsion develops.

5. NON-UNIFORM TORSION

Restraining the axial displacement due to warping results into a torsional warping normal elastic stresses σ_w expressed as:

$$\sigma_w = \mathrm{E}\varepsilon_z = E\frac{\partial u}{\partial z} \tag{3}$$

This induces secondary shear stresses that satisfy the equilibrium condition:

$$\frac{\partial \sigma_w}{\partial z} + \frac{\partial \tau_w}{\partial s} = 0 \tag{4}$$

These shear stresses result in a secondary shear flow: $q_w = \tau_w t$, which in turn induces a secondary torsional moment, also known as Wagner's torsional moment T_w.

$$T_w = \int_A q_w r ds = -E\frac{d^3\theta}{dz^3}\int_A \omega^2 t ds = -EI_w\frac{d^3\theta}{dz^3} \tag{5}$$

where the torsional warping constant I_w is defined as:

$$I_w = \int_A \omega^2 t ds \tag{6}$$

Introducing the torsional bimoment, or torsional warping moment B_t or M_w as:

$$M_w = EI_w\frac{d^2\theta}{dz^2} \tag{7}$$

equation (3) can be rewritten as:

$$\sigma_w = \mathrm{E}\omega\frac{d^2\theta}{dz^2} = \frac{M_w}{I_w}\omega \tag{8}$$

Finally the differential equation of non uniform torsion is expressed as:

$$EI_w\frac{d^3\theta}{dz^3} - GK_t\frac{d\theta}{dz} = -T = -(T_s + T_w) \tag{9}$$

where T is the sum of the St Venant's torsion T_s and Wagner's torsion T_w.

6. DISTORTION

The eccentric loading of Figure 3 is decomposed into the symmetric loading of Figure 3(a) and the anti symmetric loading of Figure 3(b).

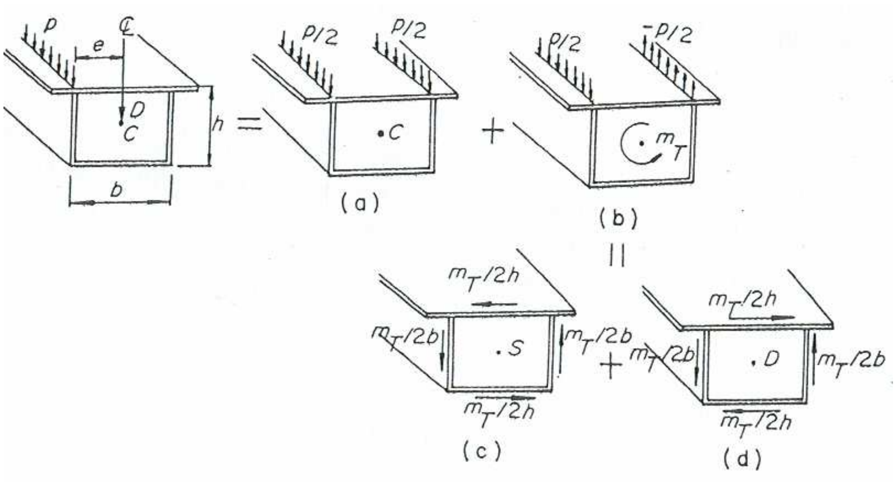


Figure 3. Decomposing the load into symmetric and anti symmetric loading

The symmetric loading results in bending of the beam, while the anti symmetric loading could be further analyzed as in Figure 3(c) and Figure 3(d). Figure 3(c) shows the shear stresses invoked by non uniform torsion while Figure 3(d) shows stresses developed so that state (b) is equal to the sum of states (c) and (d). The stresses shown in Figure 3 (d) correspond to distortion. Distortion deforms the box section and results in angular distortion Θ of the corners of the box. This distortional angle (Figure 4.) is defined by the geometrical relation:

$$\Theta = \frac{v_l - v_u}{h} + \frac{w_i - w_0}{b} \tag{10}$$

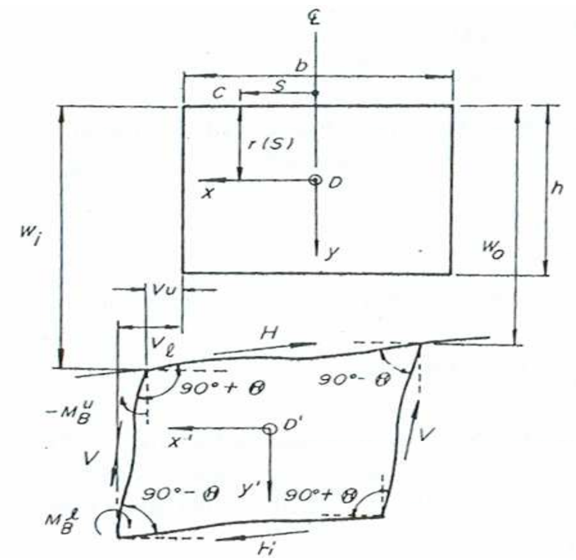


Figure 4. Distortion of the Rectangular Box section

Deformation of the box sections results in distortional warping:

$$u(z,s) = \frac{\partial\Theta(z)}{\partial z}\omega_D(s), \quad \text{where} \quad \omega_D = -\int_0^s r_d(s)ds \tag{11}$$

and $r_d(s)$ is the distance from the distortional center D.
With respect to non uniform torsion, restrain of the axial displacement leads to distortional normal stresses σ_{Dw} given as:

$$\sigma_{Dw} = E\varepsilon = E\frac{\partial u}{\partial z} = E\frac{\partial^2\Theta}{\partial z^2}\omega_D \tag{12}$$

Similarly, to equilibrate these axial stresses distortional shear stresses τ_{Dw} are developed as determined by:

$$\frac{\partial\sigma_{Dw}}{\partial z} + \frac{\partial\tau_{Dw}}{\partial s} = 0 \tag{13}$$

Moreover, with respect to non uniform torsion the distortional bimoment B_d or M_{Dw} is defined as:

$$M_{Dw} = EI_{Dw}\frac{d^2\Theta}{dz^2} \tag{14}$$

where I_{Dw} is the distortional warping constant defined as: $I_{Dw} = \int_A \omega_D^2 dA$

Therefore, the normal stresses can now be rewritten as:

$$\sigma_{Dw} = \frac{M_{Dw}}{I_{Dw}} \omega_D \qquad (15)$$

In addition the shear stresses result in a distortional shear flow:

$$q_{Dw} = \tau_{Dw} t \qquad (16)$$

and the distortional moment T_{Dw} can be given as:

$$T_{Dw} = -\frac{\partial M_{Dw}}{\partial z} = -EI_{Dw} \frac{\partial^3 \Theta}{\partial z^3} \qquad (17)$$

As the box section deforms due to distortion Θ, it behaves like a plane frame of unit width with its members acting as beams under bending. The stiffness this frame develops against distortion can be defined as K_{Dw}:

For a single cell box section is expressed as:

$$K_{Dw} = \frac{24EI_{Web}}{a_0 h} \qquad (18)$$

where,

$$a_0 = 1 + \frac{2b/h + 3(I_u + I_l)/I_h}{(I_u + I_l)/I_h + (6h/b)(I_u I_l / I^2{}_h)}, \quad I_u = \frac{t_u^3}{12(1-v^2)}, \quad I_h = \frac{t_w^3}{12(1-v^2)}, \quad I_l = \frac{t_l^3}{12(1-v^2)}$$

The differential equation for distortion is determined as follows:

$$EI_{Dw} \frac{d^4 \Theta}{dz^4} + K_{Dw} \Theta = \frac{m_T}{2} \qquad (19)$$

$$\frac{d^4 \Theta}{dz^4} + 4a_2^4 \Theta = \frac{1}{\mathrm{EI}_{Dw}} \frac{m_T}{2} \quad \text{with} \quad a_2 = \sqrt[4]{\frac{K_{Dw}}{4EI_{Dw}}} \qquad (20)$$

which is similar to the equation of a beam on elastic foundation.

7. MULTICELL BEAM ELEMENT

The classical beam theory does not account for non uniform torsion and distortional effects. Therefore, an one-dimensional beam theory is formulated for multicell thin-walled elements based on the above theory having 9 degrees of freedom at each end. Obtaining a such type of element, torsional warping and distortion can be considered.

Having 9 degrees at each end the element has a total of 18 degrees of freedom. The vectors of end displacements $\{d\}$ and nodal forces $\{f\}$ are defined as:

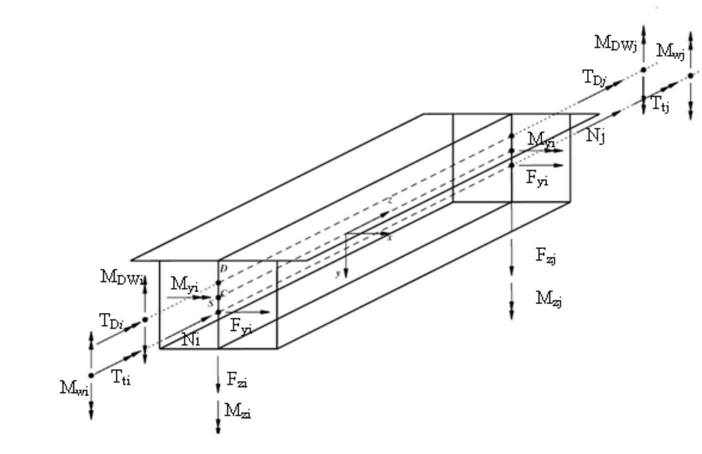


Figure 5 Nodal Forces

$$\{d\}^T = \{u_{0i}\ \mathrm{v}_{0i}\ \mathrm{w}_{0i}\ \theta_i\ \mathrm{v}'_{0i}\ -w'_{0i}\ -\theta'_i\ \Theta_i\ -\Theta'_i\ u_{0j}\ \mathrm{v}_{0j}\ \mathrm{w}_{0j}\ \theta_j\ \mathrm{v}'_{0j}\ -w'_{0j}\ -\theta'_j\ \Theta_j\ -\Theta'_j\} \quad (21)$$

$$\{f\}^T = \{N_{zi}\ \mathrm{F}_{xi}\ \mathrm{F}_{yi}\ \mathrm{T}_{ti}\ \mathrm{M}_{xi}\ \mathrm{M}_{yi}\ \mathrm{M}_{wi}\ \mathrm{T}_{Di}\ M_{Dwi}\ N_{zj}\ \mathrm{F}_{xj}\ \mathrm{F}_{yj}\ \mathrm{T}_{tj}\ \mathrm{M}_{xj}\ \mathrm{M}_{yj}\ \mathrm{M}_{wj}\ \mathrm{T}_{Dj}\ M_{Dwj}\} \quad (22)$$

where:

$$T_t = T_s + T_w \quad (23)$$

8. GOVERNING DIFFERENTIAL EQUATIONS

The differential equations that describe the problem are the following:

$$-EAu_0'' = q_z \quad (24)$$

$$EI_x \frac{d^4 w_0}{dz^4} = q_y + m_x' \quad (25)$$

$$EI_y \frac{d^4 v_0}{dz^4} = q_x - m_y' \quad (26)$$

$$EI_w \frac{d^4\theta}{dz^4} - GK_T\theta'' = m_T + m_w' + y_s q_x - y_s m_y' \quad (27)$$

$$EI_{Dw} \frac{d^4\Theta}{dz^4} + K_{Dw}\Theta = \frac{1}{2} m_T \quad (28)$$

where $u_{0,}\ v_{0,}\ w_0$ are the axial, transverse and vertical components of displacement of the centroid. Using the above equations and appropriate shape functions, determined using the analytical expressions of the homogeneous solutions, the (18x18) stiffness matrix is derived. Bending stiffness coefficients of the 3-D element are the same as in classical beam theory with the exception of the torsional degrees of freedom. The new torsional and distortional degrees of freedom are calculated using appropriate relationships. In addition, bending, torsional and distortional entries are uncoupled and the stiffness coefficients are determined using closed form expressions.

9. Eurocode 3, Part 2

Eurocode 3 specifications for non uniform torsion and distortion are listed in paragraph 6.2.1. More specifically they require:

(1) For members subject to torsion, both torsional and distortional effects should be taken into account.

(2) Where the effects of transverse stiffness in the cross section or of diaphragms that are built in to reduce distortional deformations shall be determined, the combined effect of bending, torsion and distortion may be analyzed with an appropriate elastic model for the members.

(3) Distortional effects may be disregarded in the member where due to the transverse bending stiffness in the cross section and/or diaphragm action, the effects from distortion do not exceed 10% of the bending effects.

Therefore normal stresses induced by torsional and distortional warping must be limited to 10% of bending normal stresses, otherwise these phenomena can not be disregarded. In other words if this ratio is exceeded, analysis employing classical beam theory is not considered accurate.

10. THE USE OF DIAPHRAGMS

Consider a beam of length L subjected to a vertical uniform load q and a uniform torsional an distortional load m_t, m_d respectively. Torsion is considered fixed at both ends. This beam has diaphragms only at its ends.

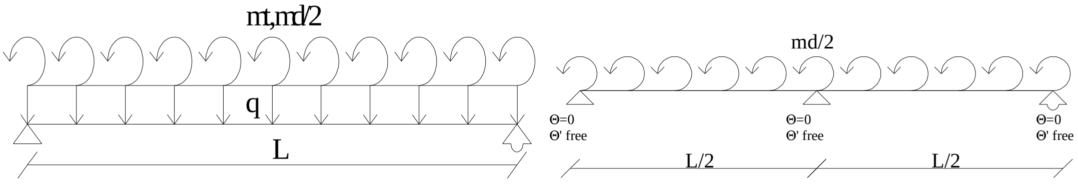
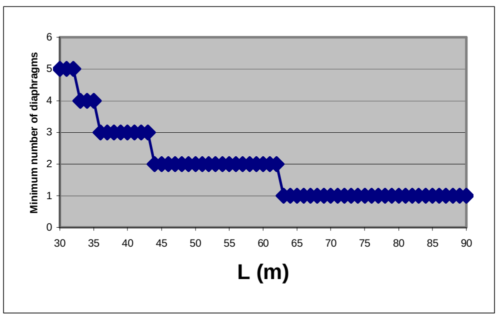


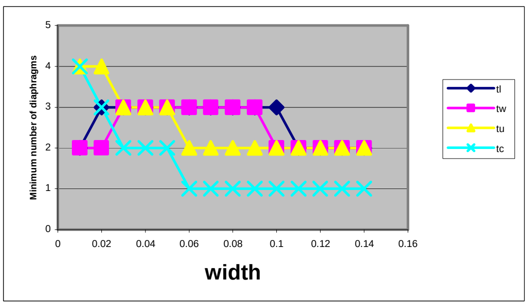
Figure 6. Simply Supported Beam without and with Intermediate Diaphragm

The boundary conditions are: $w_i = 0,\ w_i'' = 0,\quad \theta_i = 0,\quad \Theta_i = 0 \quad i = 1,2$. If a diaphragm is placed in the middle of the span the boundary conditions for bending or torsion will not be affected. Thus the induced normal stresses remain the same. For distortional boundary conditions, distortional warping at the midpoint of the span remains free. However the diaphragm restrains the distortion Θ of the midspan and since the transverse stiffness of the diaphragm is significantly higher than the one of the beam, Θ can be considered fully fixed. Restraining distortion Θ in an interior point results to decrease of the maximum values of distortional normal stress. Further more the use of 2 interior diaphragms at L/3 leads to further increase of distortional normal stresses. By increasing the number of interior diaphragms distortional normal stresses reduce, while normal stresses due to torsion and bending remain the same. Therefore, the ratio of bending to torsional and distortional normal stresses reduces too. Consequently, for every box girder a minimum number of diaphragms can be determined so that the ratio is under 10% and classical beam elements can be used in the analysis providing results that may differ less than 10% from the more accurate analysis.

11. NUMERICAL EXAMPLES

Eurocode 1 specifies the number of lanes, their width and traffic loads for various categories of roads. Moving loads are placed in a way that produces maximum normal bending stresses. In addition, eccentric loads are placed in the most unfavorable position to produce maximum torsional and distorsional stresses. Consider a cross section with b=10m, c=4m, h=3m, t_u=0.06m, t_l=0.06m, t_c=0.05m, t_w=0.07m. The material of the section is steel, E=200 GPa, ν=0.3. For this section the required number of diaphragms is determined for lengths varying from 20 to 90 m.





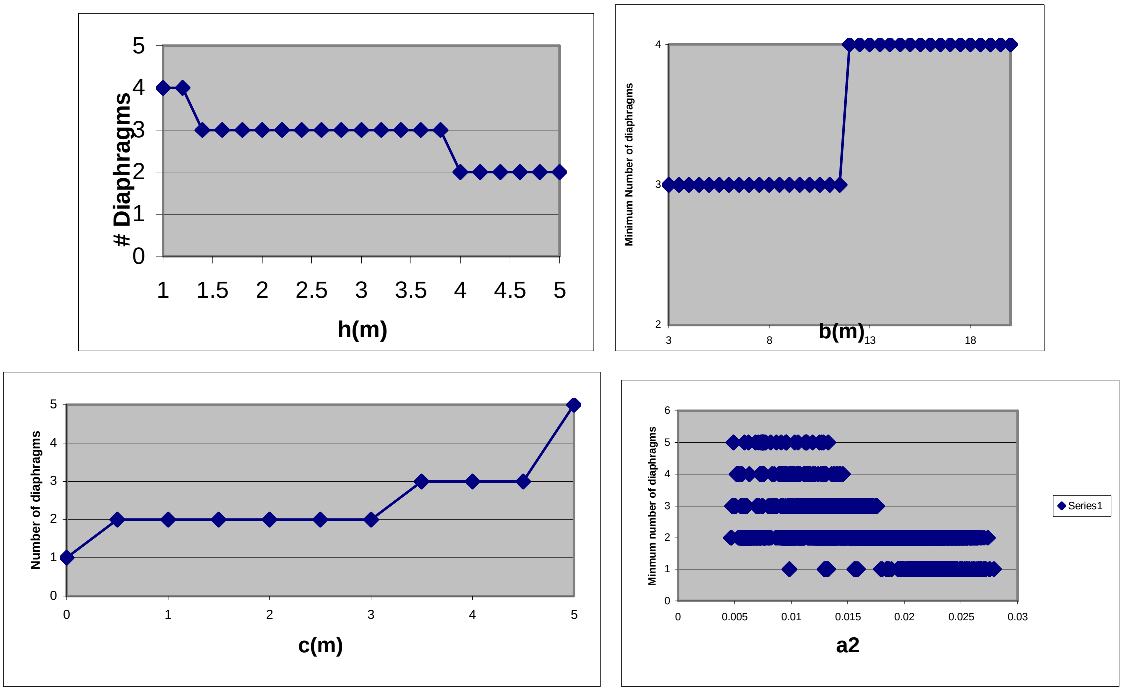


Figure 7. Variation of Number of Diaphragms with Key Parameters

Notice that the required number of diaphragms reduces as the length increases. Torsional and distortional normal stresses increase as the length increases, however at a slower rate than the ones induced by bending. Therefore the smaller the span the greater the number of the diaphragms, despite of what one would expect. Increasing the thickness of different parts of the box section doesn't signify a single trend for the number of diaphragms. The sum of distortional and torsional normal stresses reduces more rapidly than those induced by bending as the height of the box section increases. Because the loads are defined in traffic lanes bending and torsional and distortional uniform loads will increase as the length of the box increases. Increase of the length of the beam leads to an increased number of required diaphragms and this is anticipated as the lever arm of the eccentric load increases. Similar behaviour is observed for the increase of the cantilever length. It is apparent that all nine parameters affect the number of diaphragms required. And the maximum influence of these parameters depends on the values of the other parameters, with the exception of the length which when increased leads to a decrease of the number of diaphragms. In the literature the parameter $a_2 = \sqrt[4]{\dfrac{K_{Dw}}{4 * E * I_{dw}}}$ is used to describe the problem. However by giving 10 different values from 0.01 to 0.014 for t_u, t_l, t_w and t_c one gets the last diagram of Figure 7. For sections with the same value of a_2, but different values of t_u, t_l, t_w, $t_c$the required number of diaphragms may differ significantly. This shows that a_2 cannot be considered as the sole parameter that determines the number of diaphragms.

12. CONCLUSIONS

A general thin-walled closed section beam element is formulated that accounts for non uniform torsion and distortion of the cross-section. The element is exact in the sense that satisfies the governing equations and as such is accurate for every length. The equivalent nodal loads determined for different types of distributed loading are developed. Based on this element the minimum number of diaphragms needed to reduce stresses due to non uniform torsion and distortion below the threshold of the 1/10 of the bending stresses is determined and the variation with respect of the eight parameters that affect the problem is demonstrated. As shown from the examples the minimum number of diaphragms is a function of all the eight parameters of the problem. This is why a practical rule for diaphragm spacing cannot be issued.

Moreover, the use of fewer diaphragms, than the ones required, signifies that the analysis doesn't comply with the demands of the Eurocode 3. Even more, torsional and distortional normal stress, that are developed in the section, are ignored although they consist a significant portion of the ones induced by bending. In some cases, with not proper use of diaphragms, the distortional stresses have maximum values significantly bigger than bending normal stresses. Finally, there are also cases where in order to keep the ratio of stresses under 10 % the distance between diaphragms is extremely small and the design is not feasible. In these cases either the cross section should be changed or one should perform an exact analysis taking into account torsion and distortion.

Therefore it is strongly recommended that one follows the required number of diaphragms proposed after exact analysis, as in this work. The minimum number of diaphragms for each case of girder described by the values H, b, t_w, t_l, t_u, c, t_c, L can be acquired by the nine dimensioned matrix stored in the internet page: http:\\users.ntua.gr/vkoum

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ΣΤΟΙΧΕΙΟ ΔΟΚΟΥ ΛΕΠΤΟΤΟΙΧΗΣ ΚΛΕΙΣΤΗΣ ΔΙΑΤΟΜΗΣ ΜΕ ΑΝΟΜΟΙΟΜΟΡΦΗ ΣΤΡΕΨΗ ΚΑΙ ΔΙΑΣΤΡΟΦΗ ΤΗΣ ΔΙΑΤΟΜΗΣ

Εμμανουήλ Ν. Χατζής, Σάββας Π. Τριανταφύλλου και Βλάσης Κ. Κουμούσης

Εργαστήριο Στατικής & Αντισεισμικών Ερευνών
Εθνικό Μετσόβιο Πολυτεχνείο
Πολυτεχνειούπολη ζωγράφου, GR-15773, ΑΘΗΝΑ
e-mails: manolis72003@hotmail.com, savtria@hahoo.gr, vkoum@central.ntua.gr

ΠΕΡΙΛΗΨΗ

Οι μονοκύψελες και πολυκύψελες κιβωτιοειδείς διατομές υποβαλλόμενες σε ανεμπόδιστη στρέψη αναπτύσσουν στρέβλωση με αποτέλεσμα την ανάπτυξη ανομοιόμορφης στρέψης και διαστροφής δηλ. την παραμόρφωση της διατομής στο επίπεδό της που είναι έντονο ιδιαίτερα στη περίπτωση λεπτότοιχων μεταλλικών διατομών. Αντίστοιχα όταν εμποδίζεται η στρέβλωση αναπτύσσονται ορθές τάσεις κατά τον διαμήκη άξονα του στοιχείου, αλλά και διατμητικές τάσεις για την αποκατάσταση της ισορροπίας. Αντικείμενο της εργασίας αποτελεί η ανάπτυξη ενός ραβδωτού στοιχείου δοκού που εμπλουτίζει το στοιχείο της δοκού στο χώρο με πρόσθετους βαθμούς ελευθερίας ώστε να περιλάβει την συμπεριφορά της ανομοιόμορφης στρέψης και της διαστροφής της διατομής. Το στοιχείο αυτό αναπτύσσεται με βάση την θεωρία των πεπερασμένων στοιχείων εισάγοντας ως συναρτήσεις σχήματος τις ακριβείς λύσεις των ομογενών καταστατικών εξισώσεων του προβλήματος. Το παραγόμενο με βάση την αρχή των δυνατών έργων μητρώο δυσκαμψίας διαστάσεων 18x18 και οι προκύπτουσες ισοδύναμες ακραίες δράσεις, για τις συνήθεις κατανεμημένες φορτίσεις, είναι ακριβείς κατά την έννοια της μητρωϊκής Στατικής και επιτρέπουν την χρήση στοιχείων οιουδήποτε μήκους. Οι διάφοροι κανονισμοί μεταξύ των οποίων και ο Ευρωκώδικας 3 επιτρέπουν την απλοποιητική χρήση στοιχείων δοκού μόνο όταν οι ορθές τάσεις λόγω της ανομοιόμορφης στρέψης και της διαστροφής της διατομής δεν ξεπερνούν ένα μικρό ποσοστό των καμπτικών τάσεων, συνήθως της τάξεως του 10%. Έτσι κατά την ανάλυση, ή θα πρέπει να χρησιμοποιηθεί το προτεινόμενο στοιχείο και οι ισοδύναμες ακραίες δράσεις του ή θα πρέπει να προβλεφθεί η χρήση ενδιάμεσων διαφραγμάτων που περιορίζουν τις πρόσθετες τάσεις των φαινομένων αυτών στα όρια που ορίζουν οι κανονισμοί. Έτσι για δεδομένη λεπτότοιχη διατομή και μήκος ανοίγματος προκύπτει, μέσω της ακριβούς ανάλυσης με τα στοιχεία πολυκύψελης διατομής, ένας ελάχιστος αριθμός απαιτούμενων διαφραγμάτων. Ο αριθμός αυτός εξαρτάται από οκτώ παραμέτρους και ως εκ τούτου καθορίζεται σε πινακοποιημένη μορφή. Στην εργασία παρουσιάζεται η μεταβολή του ελάχιστου αριθμού απαιτούμενων ενδιάμεσων διαφραγμάτων με βάση την μεταβολή των κύριων παραμέτρων, με τις υπόλοιπες θεωρούμενες σταθερές, οι δε πλήρεις πίνακες παρέχονται στην ηλεκτρονική διεύθυνση: http:\\users.ntua.gr/vkoum.