# STEEL BEAM-COLUMNS ON ELASTIC FOUNDATION: COUPLED INSTABILITY MODES

**Dimitris S. Sophianopoulos**
Civil Engineer PhD, P.E., Assistant Professor
Department of Civil Engineering, University of Thessaly
Volos, Greece
e-mail: dimsof@civ.uth.gr, dimsoph@otenet.gr

**A. Sbarounis**
Civil Engineer P.E., Teaching Assistant (Retired)
Metal Structures Lab., Department of Civil Engineering, National Technical University
Athens, Greece
e-mail : sbarouni@central.ntua.gr

**K. S. Papachristou**
Civil Engineer MSc, P.E., PhD Candidate
Department of Civil Engineering, University of Thessaly
Volos, Greece
e-mail : kpapach@uth.gr

## 1. SUMMARY

We present the linear as well as harmonic nonlinear theoretical background of the postbuckling response of steel beam-columns on Winkler-type elastic foundation. Preliminary results in very good agreement with existing ones indicate coupled remote singularities and mode jumping. Further study is well under way, being the subject of the 3rd author's PhD Thesis.

## 2. INTRODUCTION

In the context of nonlinear stability and bifurcation theory, phenomena associated with coupled instabilities, remote postcritical branching behavior and mode jumping of real continuous structural elements (such as beams, plates an shells) are of immense importance for design purposes and everyday engineering practice. More specifically, the postbuckling response of axially compressed steel (elastic) beams resting on an elastic foundation of Winkler type, which can be used as a model for the simulation of the buckling behavior of axially compressed thin circular cylindrical shells, is also related to practical applications (welded railway tracks, stability of the top chord of low-truss bridges etc.) and constitutes the main objective of the present work. Focusing on cases of mode coupling, and employing initially a linearized and moreover a nonlinear harmonic analysis some preliminary results are presented, indicating sudden changes in the postbuckling patterns. The whole scientific matter is under further study, and higher order singularities are anticipated, giving rise to complicated phenomena in remote postcritical domains.

## 3. PROBLEM STATEMENT AND MATHEMATICAL ANALYSIS

We consider an initially perfect, elastically supported steel beam of length L and uniform cross-section, of flexural rigidity EI, laterally supported on a Winkler-type elastic foundation and acted upon by an axially compressive loading P. Denoting by $k_f$ the elastic foundation modulus, by W the transverse deflection and employing a more "exact" expression for the curvature, the differential equation governing the equilibrium in the deformed state, in dimensionless form, is given by [1]

$$w'''' + \lambda w'' + \mu w = -\frac{1}{2}\left(w''''w'^2 + 6w'''w''w' + 2w''^3\right) \quad , \quad \left(0 \le x \le 1\right) \tag{1}$$

where $x = \frac{X}{L}$ , $w = \frac{W}{L}$ , $\lambda = \frac{PL^2}{EI}$ , $\mu = \frac{k_f L^4}{EI}$. For a simply supported beam, as the one depicted in Fig.1, the corresponding boundary conditions are

$$-EIw''(1+\frac{1}{2}w'^2) = 0 \text{ at } x = 0,1 \tag{2}$$

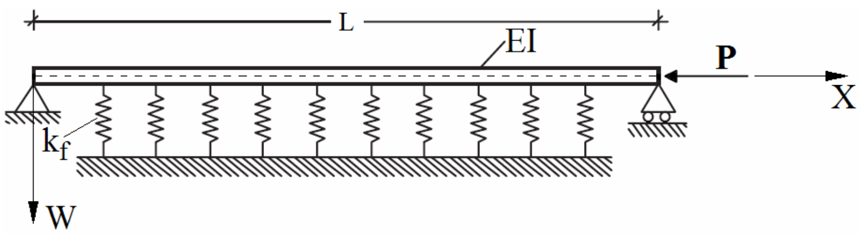


*Fig. 1 Simply supported beam-column on an elastic Winkler foundation*

Eqs.(1) and (2) constitute a highly nonlinear, non-homogeneous two-point boundary value problem, which will be addressed in what follows.

### 3.1 Linear buckling analysis

For small values of w the nonlinear terms in eq.(1) can be neglected, i.e.

$$w'''' + \lambda w'' + \mu w = 0 \tag{3}$$

Seeking a solution of the form $w = Ae^{\rho x}$ we reach to the characteristic equation

$$\rho^4 + \lambda\rho^2 + \mu = 0 \tag{4}$$

which for $\lambda \ge 2\sqrt{\mu}$ admits two pairs of pure imaginary roots

$\rho_{1,2} = \pm ik$ , $\rho_{3,4} = \pm i\bar{k}$ , $i=\sqrt{-1}$ (5)

with the following characteristics:

$$k = \sqrt{\frac{\lambda}{2} - \sqrt{\frac{\lambda}{2} - \mu}} \quad , \quad \bar{k} = \sqrt{\frac{\lambda}{2} + \sqrt{\frac{\lambda}{2} - \mu}} \tag{6a}$$

$$\bar{k}^2 - k^2 = 2\sqrt{\frac{\lambda^2}{4} - \mu} \geq 0 \ , \ \bar{k}^2 k^2 = \mu \ , \ \frac{k}{\bar{k}} = \frac{\lambda}{2\sqrt{\mu}} - \sqrt{\frac{\lambda^2}{4\mu} - 1} \tag{6b}$$

It should be noted that $k$, $\bar{k}$ are independent of the boundary conditions, while it is evident that for $\lambda \geq 2\sqrt{\mu}$ there are two periodic (harmonic) solutions of the linearized eq.(3). If the ratio of the corresponding wave lengths is a rational number modulated periodicity arises, while if this is not the case quasi-periodicity will result. The general solution of eq.(3) is:

$$w = q_m \sin kx + q_n \sin \bar{k}x + \bar{q}_m \cos kx + \bar{q}_n \cos \bar{k}x \tag{7}$$

where $q_m, q_n, \bar{q}_m, \bar{q}_n$ are real numbers that can be determined via the boundary conditions. For the simply supported beam these conditions are $w(0) = w''(0) = w(1) = w''(1) = 0$; hence only two sinusoidal terms appear ($q_m$ and $q_n$) and the buckling equation yields

$$\left(\bar{k}^2 - k^2\right) \sin k \sin \bar{k} = 0 \tag{8}$$

$$\text{If } q_n = 0 \Rightarrow \sin k = 0 \Rightarrow k = m\pi \ , \ m \in \mathbf{N}$$
$$\text{If } q_m = 0 \Rightarrow \sin \bar{k} = 0 \Rightarrow \bar{k} = n\pi \ , \ n \in \mathbf{N} \tag{9}$$
$$\text{If } q_m q_n \neq 0 \Rightarrow \sin k \sin \bar{k} = 0$$

The last of eqs.(9) yields also roots of the form $\sin k = \sin \bar{k} = 0$, implying $k = m\pi$ and $\bar{k} = n\pi$, where $m, n \in \mathbf{R}$ representing the number of semi-waves of the buckling mode. This 3rd case implies the coupling of two sinusoidal waveforms with different amplitudes, provided that $k/\bar{k} = m/n$ is a rational number. Thus it is concluded that for the simply supported beam quasi-periodicity is impossible. Solving the characteristic equation (4) for $\rho = m\pi$, after some elaboration, we get

$$\frac{\lambda}{\sqrt{\mu}} = \frac{m^2\pi^2}{\sqrt{\mu}} + \frac{\sqrt{\mu}}{m^2\pi^2} = \left(\frac{m\pi}{\sqrt[4]{\mu}}\right)^2 + \left(\frac{\sqrt[4]{\mu}}{m\pi}\right)^2 \tag{10}$$

where

$\frac{\lambda}{\sqrt{\mu}} = \frac{Pl^2/EI}{\sqrt{K_f l^4/EI}} = \frac{P}{\sqrt{K_f/EI}}$ is the loading parameter and $\frac{\sqrt[4]{\mu}}{\pi} = \frac{\sqrt[4]{K_f l^4/EI}}{\pi} = \frac{l}{\pi}\sqrt[4]{K_f/EI}$ is the length parameter. Plotting the curves $\frac{\lambda}{\sqrt{\mu}}$ vs $\frac{\sqrt[4]{\mu}}{\pi}$ for various values of m we obtain the graphic representation of Fig.2, known also from the linear buckling of plates and [2] : Evidently, in the general case the critical state is discrete, except at the intersections of two m-curves, where a compound branching point is revealed. This can only happen when $\frac{\sqrt[4]{\mu}}{\pi} = \sqrt{mn}$, with corresponding critical loads equal to $\frac{\lambda_m}{\sqrt{\mu}} = \frac{\lambda_n}{\sqrt{\mu}} = \frac{m}{n} + \frac{n}{m}$, where eq.(7) possesses solutions containing coupled sinusoidal waveforms. For the simply supported beam this case is associated with n = m +1. Since the general case of uncoupled waveforms has been recently dealt with in the literature [1], the present study focuses on the special case of coupled waveforms.

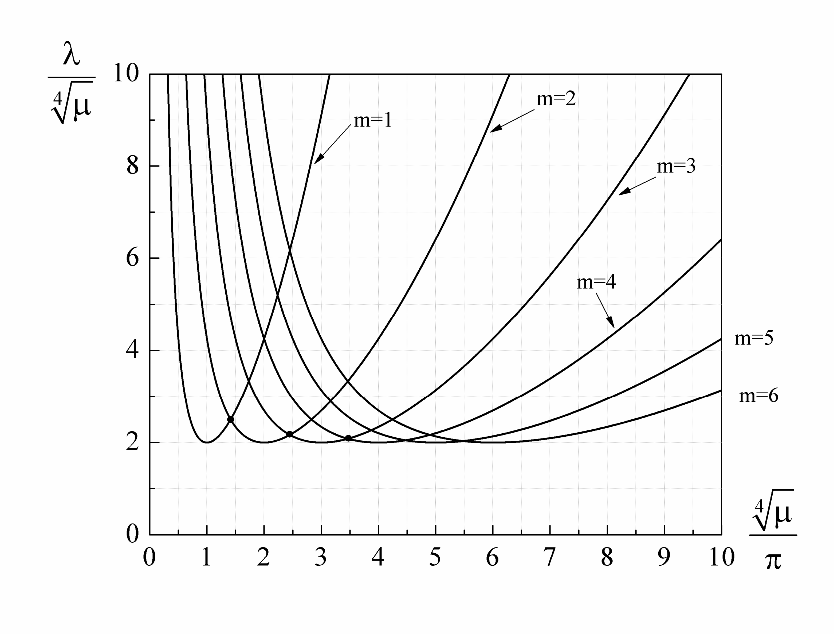


*Fig. 2 Curves of critical linear buckling loads. The circles mark compound branching points for solutions containing 1 and 2, 2 and 3 as well as 3 and 4 half-waves.*

### 3.2 Nonlinear energy formulation - Harmonic analysis

All the possible equilibrium configurations of the simply beam in the nonlinear regime, can be treated using a Fourier series representation of the deflection [3], given by

$$w = \sum_{a=1}^{\infty} q_a \sin a\pi x = q_a \sin a\pi x \tag{11}$$

where the Einstein summation convention is adopted, with $a \in \mathbf{N}$. Differentiation of the above expression gives

$$w_{,x} = q_a a\pi \cos a\pi x \tag{12a}$$

$$w_{,xx} = -q_a a^2\pi^2 \sin a\pi x \tag{12b}$$

$$w_{,x}^2 = q_a q_b ab\pi^2 \cos a\pi x \cos b\pi x \tag{12c}$$

$$w_{,x}^4 = q_a q_b q_c q_d abcd\pi^4 \cos a\pi x \cos b\pi x \cos c\pi x \cos d\pi x \tag{12d}$$

$$w^2 = q_a q_b \sin a\pi x \sin b\pi x \tag{12e}$$

$$w_{,xx}^2 = q_a q_b a^2 b^2 \pi^4 \sin a\pi x \sin b\pi x \tag{12f}$$

$$w_{,xx}^2 w_{,x}^2 = q_a q_b q_c q_d a^2 b^2 cd\pi^6 \sin a\pi x \sin b\pi x \cos c\pi x \cos d\pi x \tag{12g}$$

$$w_{,xx}^2 w_{,x}^4 = q_a q_b q_c q_d q_e q_f a^2 b^2 cdef\pi^8 \sin a\pi x \sin b\pi x \cos c\pi x \cos d\pi x \cos e\pi x \cos f\pi x \tag{12h}$$

where the subscript following the comma denotes differentiation with respect to x.
Performing a Taylor expansion on the system's total potential energy function in the undeformed state, one may write:

$$V(q_a, \lambda) \equiv U - \lambda\Omega =$$
$$\frac{1}{2!}U_{,ab}(0)q_a q_b + \frac{1}{4!}U_{,abcd}(0)q_a q_b q_c q_d + \frac{1}{6!}U_{,abcdef}(0)q_a q_b q_c q_d q_e q_f - \frac{\lambda}{2!}\Omega_{,ab}(0)q_a q_b \tag{13}$$

where U is the elastic energy and Ω the work of the external loading. Taking into account the derivatives given in (12a-h) we get

$$U_{,ab}(0) = 0 \ \text{for} \ a \neq b, \ U_{,aa}(0) = \frac{EI}{2l}(a^4\pi^4 + \mu) \ \text{for} \ a = b, \tag{14a}$$

$$\Omega_{,ab}(0) = 0 \ \text{for} \ a \neq b, \ \Omega_{,aa}(0) = \frac{EI}{2l}a^2\pi^2 \ \text{for} \ a = b, \tag{14b}$$

$$U_{,abcd} = \frac{12EI\pi^6}{l}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\sum_{d=1}^{\infty} a^2b^2cdq_aq_bq_cq_d\int_0^1 \sin a\pi x \sin b\pi x \cos c\pi x \cos d\pi x dx \tag{14c}$$

$$U_{,abcdef} = \frac{90EI\pi^8}{l}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\sum_{d=1}^{\infty}\sum_{e=1}^{\infty}\sum_{f=1}^{\infty} a^2b^2cdefq_aq_bq_cq_dq_eq_f \times$$

$$\int_0^1 \sin a\pi x \sin b\pi x \cos c\pi x \cos d\pi x \cos e\pi x \cos f\pi x dx \tag{14d}$$

Evidently, the quadratic forms $U_{,ab}$ and $\Omega_{,ab}$ are diagonal and consequently the Fourier harmonics represent the buckling eigenmodes. The corresponding critical loads are derived from the equation

$$U_{,aa} - \lambda\Omega_{,aa} = \frac{EI}{2l}(a^4\pi^4 - \lambda a^2\pi^2 + \mu) = 0 \tag{15}$$

with $\lambda_{cr} = m^2\pi^2 + \frac{\mu}{m^2\pi^2}$ (1st buckling load for a=m)

A Taylor expansion with respect to the critical state ($\lambda_{cr}$,0) yields the potential energy functional

$$W = \frac{1}{2!}(U_{,aa} - \lambda_{cr}\Omega_{,aa})q_a^2 + \frac{1}{4!}U_{,abcd}q_aq_bq_cq_d + + \frac{1}{6!}U_{,abcdef}q_aq_bq_cq_dq_eq_f - \frac{\lambda - \lambda_{cr}}{2!}\Omega_{,aa}q_a^2 \tag{16}$$

the derivatives of which are equal to:

$$W^c_{,ab} = U_{,ab}(0) - \lambda_{cr}\Omega_{,ab}(0) = 0 \ \text{for} \ a \neq b \ , \ W^c_{,mm} = U_{,mm}(0) - \lambda_{cr}\Omega_{,mm}(0) = 0 \tag{17a,b}$$

$$W^c_{,aa} = U_{,aa}(0) - \lambda_{cr}\Omega_{,aa}(0) = \frac{EI}{2l}(a^4\pi^4 + \frac{\mu}{\pi^4})(1 - \frac{a^2}{m^2}) \ \text{for} \ a \neq m \tag{17c}$$

$$W'^c_{,ab} = -\Omega_{,ab}(0) = 0 \ \text{for} \ a \neq b \ , \ W'^c_{,mm} = -\Omega_{,mm}(0) = -\frac{EI}{2l}m^2\pi^2 \tag{17d,e}$$

$$W^c_{,aaaa} = \frac{3EI}{2l}a^6\pi^6 \ , \ W^c_{,baaa} = 0 \ \text{for} \ b \neq 3a \ , \ W^c_{,baaa} = \frac{27EI}{2l}a^6\pi^6 \ \text{for} \ b = 3a \tag{17f-h}$$

$$W^c_{,bbaa} = 3EIa^2b^4\pi^6 \ , \ W^c_{,cbaa} = -\frac{3EI}{2l}a^2b^2(2a-b)^2 \ \text{for} \ c = 2a - b > 0 \tag{17i,j}$$

$$W^c_{,cbaa} = \frac{3EI}{2l}a^2b^2(2a+b)^2 \ \text{for} \ c = 2a + b \ , \tag{17k}$$

$$W^c_{,cbaa} = \frac{3EI}{2l}a^2b^2(2a-b)^2 \ \text{for} \ c = -2a + b > 0 \tag{17l}$$

$$W^c_{,cbba} = -\frac{3EI}{2l}ab^3(a-2b)^2 \ \text{for} \ c = a - 2b > 0 \tag{17m}$$

$$W^c_{,cbba} = \frac{3EI}{2l}ab^3(a+2b)^2 \ \text{for} \ c = a + 2b \tag{17n}$$

$$W^c_{,cbba} = \frac{3EI}{2l}ab^3(a-2b)^2 \ \text{for} \ c = -a + 2b > 0 \tag{17o}$$

$$W^c_{,ccba} = -\frac{3EI}{32l}ab(a+b)^4 \ \text{for} \ c = \frac{a+b}{2} \tag{17p}$$

$$W^{c}_{,ccba} = \frac{3EI}{32l} ab(a-b)^4 \text{ for } c = \pm\frac{a-b}{2} > 0 \quad (17q)$$

$$W^{c}_{,dcba} = -\frac{3EI}{2l} abc^2d^2 \text{ for } d = a-b-c \text{ or } d = a+b-c \text{ or } d = -a+b-c \quad (17r)$$

$$W^{c}_{,dcba} = \frac{3EI}{2l} abc^2d^2 \text{ for } d = a-b+c \text{ or } d = a+b+c \text{ or } d = -a+b+c \text{ or } d = -a-b+c \quad (17s)$$

$$W^{c}_{,aaaaaa} = \frac{45}{8l} a^8\pi^8 \text{ , } W^{c}_{,baaaaa} = \frac{135}{16l} a^6b^2\pi^8 \text{ for } b = 3a \text{ , } W^{c}_{,baaaaa} = \frac{45}{16l} a^6b^2\pi^8 \text{ for } b = 5a \quad (17t\text{-}v)$$

In all the above expressions the prime represents differentiation with respect to λ. Since cubic and 5th order terms are absent, the energy quadratic form W can be set in diagonal form. Hence, there is no energy term containing more than six components and every term of special interest (i.e. of the sixth order) can be determined as the sum of six waves:

$$w = q_m \sin m\pi x + q_n \sin n\pi x + q_p \sin p\pi x + q_r \sin r\pi x + q_s \sin s\pi x + q_t \sin t\pi x \quad (18)$$

where $m, n, p, r, s, t \in \mathbf{N}$, while it is assumed that $m < n < p < r < s < t$. The full energy function for the entire set of terms contained in relations (17) can thus be built from different sets of six waves.

## 4. PRELIMINARY RESULTS

After the theoretical analysis outlined above, a procedure similar to the one adopted in the recent literature [3] is used for the rather simple case of an equilibrium solution containing the following two modes

$$w = q_m \sin m\pi x + q_n \sin n\pi x \quad (19)$$

where $m < n$ in the vicinity of the compound bifurcation point of Fig.2 corresponding to the interaction of m=3 and n=4 half-waves. Using $q_m$ and $q_n$ as generalized coordinates, we obtain the 3D representation of the equilibrium paths shown in Fig.3, being in very good agreement with existing ones of the relevant literature [3,4].

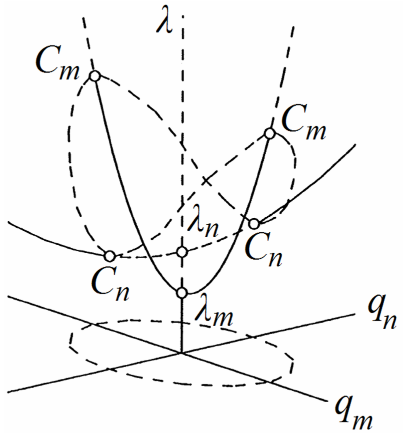


*Fig.3 Equilibrium paths of energy function (16). Stable paths are shown as solid lines and unstable paths as dashed lines.*

Buckling into m=3 half-waves occurs at the critical load $\lambda_m$, whereupon the system follows the stable uncoupled path ($q_n$=0). The uncoupled path with n=4 half-waves intersects the fundamental path at $\lambda_n > \lambda_m$, being therefore initially unstable with respect to $q_m$. A coupled solution path, with both $q_m$ and $q_n$ non-zero, intersects the lower and upper uncoupled paths at the secondary branching points $C_m$ and $C_n$ respectively, and its projection on the base plane is a closed loop. The lower uncoupled path thus loses stability at $C_m$, where under dead loading conditions a dynamic jump takes place to the upper uncoupled path ($q_m$=0), which has stabilized at $C_n$. This is a typical case of mode jumping from m=3 to n=4 half-waves. Solutions obtained at different locations on Fig.3 are depicted in Fig.4.

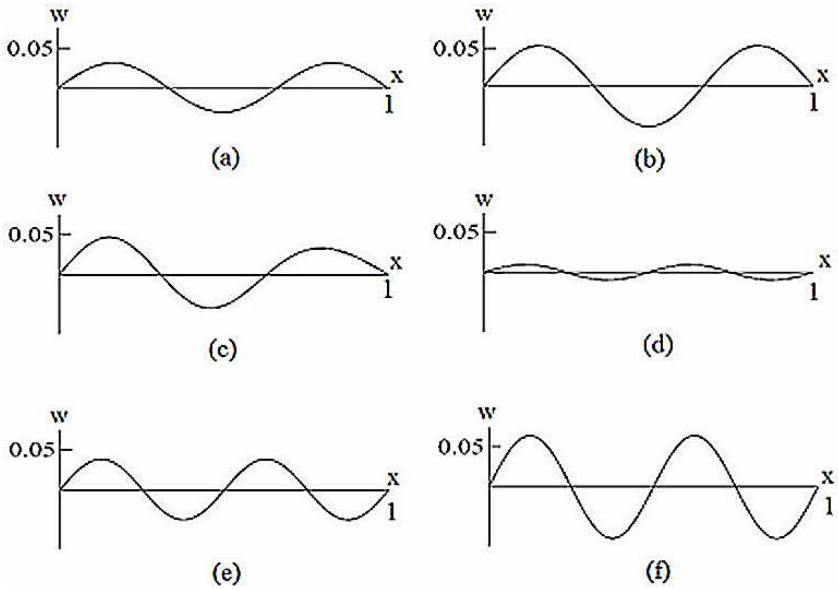


*Fig.4 Solutions at different locations on Fig.3: (a) 1st stable path; (b) $C_m$; (c) unstable path between $C_m$ and $C_n$; (d) $C_n$; (e),(f) second stable path.*

Much more complicated singularity phenomena are anticipated in remote postcritical domains, associated with higher order catastrophes, if all six components of eq.(18) are to be used. This is a matter of further in depth analysis, being the subject of the PhD thesis of the 3rd author.

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# ΧΑΛΥΒΔΙΝΟΙ ΔΟΚΟΙ-ΣΤΥΛΟΙ ΕΠΙ ΕΛΑΣΤΙΚΟΥ ΕΔΑΦΟΥΣ: ΣΥΖΕΥΓΜΕΝΕΣ ΜΟΡΦΕΣ ΑΣΤΑΘΕΙΑΣ

**Δ. Σ. Σοφιανόπουλος**
Δρ. Πολιτικός Μηχανικός ΕΜΠ, Επίκουρος Καθηγητής
Τμήμα Πολιτικών Μηχανικών, Πανεπιστήμιο Θεσσαλίας
Πεδίον Άρεως, 38 334 Βόλος, Ελλάς
e-mail: dimsof@civ.uth.gr

**Α. Σμπαρούνης**
Πολιτικός Μηχανικός ΕΜΠ, τ. Βοηθός
Εργαστήριο Μεταλλικών Κατασκευών, Τμήμα Πολιτικών Μηχανικών ΕΜΠ
Αθήνα, Ελλάς
e-mail: sbarouni@central.ntua.gr

**Κ. Παπαχρήστου**
Πολιτικός Μηχανικός Π.Θ., MSc, Υποψήφιος Διδάκτωρ
Τμήμα Πολιτικών Μηχανικών, Πανεπιστήμιο Θεσσαλίας
Πεδίον Άρεως, 38 334 Βόλος, Ελλάς
e-mail: kpapach@uth.gr

## ΠΕΡΙΛΗΨΗ

Η ανάλυση ευστάθειας δοκών-στύλων επί ελαστικού εδάφους (τύπου Winkler) διαδραματίζει σπουδαίο ρόλο στην επιστήμη του Πολιτικού Μηχανικού, με πολλές εφαρμογές, όπως σε συγκολλητές σιδηροδρομικές τροχιές, σε μεταλλικές γέφυρες, σε λεπτότοιχα κελύφη κλπ. Η παρούσα εργασία έχει σαν αντικείμενο την μεταλυγισμική συμπεριφορά χαλύβδινων τέτοιων δοκών-στύλων χωρίς αρχικές ατέλειες, με έμφαση στις ιδιαίτερες εκείνες περιπτώσεις, όπου μπορεί να υπάρξουν συζευγμένες μορφές λυγισμού, δευτερεύουσες διακλαδώσεις και απότομα ιδιομορφικά άλματα. Το έντονα μη γραμμικό πρόβλημα συνοριακών τιμών δύο σημείων που διέπει την ευστάθεια των ως άνω δομικών στοιχείων αντιμετωπίζεται μέσω αρμονικής ανάλυσης επί του συνολικού δυναμικού, ορισμένα δε προκαταρκτικά αποτελέσματα βρέθηκαν σε πολύ καλή συμφωνία με ήδη υπάρχοντα άλλων ερευνητών. Περαιτέρω εις βάθος επεξεργασία απαιτείται, που αποτελεί ουσιαστικά και το αντικείμενο της Διδακτορικής Διατριβής του 3ου εκ των συγγραφέων.