ELASTIC STABILITY OF STEEL TAPERED BEAMS

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1. ABSTRACT

The elastic stability of steel columns with non-uniform cross-section of any shape, loaded by forces and moments applied at its ends or intermediate points and with (or without) initial imperfections is studied herein. The formulation presented in this paper is based on solving the general equation of the problem by the method of series procedure using the eigenshapes of the column.

2. INTRODUCTION

It is common to use steel members with non-uniform cross-sections either as columns or as distressed members of a structure with or without bending moments. There is a wide range of various structures like building frames, bridge members, masts, cranes, etc., which in order to minimize the material required, they are designed with members of non-uniform cross-sections. On the research history about these topics, see Timoshenko [1].
On the other hand, the significance of the initial imperfections was noted very early and was studied mainly experimentally by Marston, Jensen and Lilly, the studies of which were gathered by Salmon [2]. A significant study on stepped columns consisted by two pieces through the use of Galerkin equations was presented by Dimitrof [3]. There is a great number of theoretical or experimental publications on tapered or stepped columns with or without imperfections.
In the present work, non-uniform steel members with or without imperfections (of any form), loaded by axial forces (concentrically or eccentrically applied) and by moments at its ends or intermediate points are studied. The steel members may be governed by any law, they may be tapered or stepped or may be members consisting by two unequal tapered parts. The formulation presented in this paper is based on solving the general equation of the problem through a numerical procedure procedure, using the eigenshapes of the member. A plasticity failure criterion is introduced for members (especially for short ones)

that will never reach the elastic critical buckling load. Although in this paper we study only the simply supported single-span beam, it is obvious that the method can be extended to any other type of frames, using the corresponding eigenshapes. Useful diagrams are presented for the critical buckling loads and for the equilibrium paths, showing the influence of the main member's characteristics.

3. FORMULATION OF THE PROBLEM

Let us consider the column of Fig. 1, the cross-section of which varies along the x-axis, according to a given law (parabolic or tapered or stepped column etc). The considered column is loaded by forces and moments as it is shown in Fig. 1. Moreover, the considered member may have an initial imperfection $w_o(x)$.

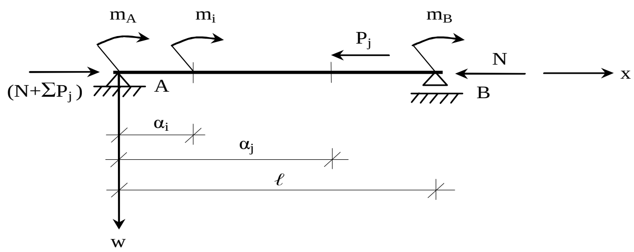

Fig. 1 Forces and moments of a column

In addition, we consider that the member is laterally supported and thus, it is protected against the possibility of buckling about the weak axis. The governing differential equation for buckling of the above column is given as follows:

$$[E\,I(x)\,w'']'' + N(w + w_o)'' + \sum_j P_j (w + w_o)''[1 - H(x - \alpha_j)] - \sum_i m_i \delta'(x - \alpha_i) = 0 \qquad (1)$$

where H(x) is the Heaviside (unit-step) function, and $\delta(x)$ is the Dirac-delta function.
The above equation (1) may be written as follows:

$$E\,Iw'''' + 2E\,I'w''' + E\,I''w'' + Nw'' + \sum_j P_j w''[1 - H(x - \alpha_j)]_o = -Nw'' - \sum_j P_j [1 - H(x - \alpha_j)] + \sum_i m_i \delta'(x - \alpha_i) \qquad (2)$$

Non-existence of moments and initial imperfection leads to a differential equation without second member, which gives the buckling critical loads, while existence of moments and initial imperfections leads to a complete differential equation, which permits us to study the equilibrium paths. We are searching a solution in the following form:

$$w(x) = c_1 X_1(x) + c_2 X_2(x) + \cdots\cdots + c_n X_n(x) \qquad (3)$$

where $X_n(x) = \sin\left(\frac{n\,\pi\,x}{\ell}\right)$, are the eigenshapes of the simply supported beam and c_n are constants under determination. Introducing eq(3) into eq(2), we get:

$$\left.\begin{aligned} & E\,I\sum_\rho c_\rho \left(\frac{\rho\,\pi}{\ell}\right)^4 \sin\frac{\rho\,\pi\,x}{\ell} - 2E\,I'\sum_\rho c_\rho \left(\frac{\rho\,\pi}{\ell}\right)^3 \cos\frac{\rho\,\pi\,x}{\ell} - E\,I''\sum_\rho c_\rho \left(\frac{\rho\,\pi}{\ell}\right)^2 \sin\frac{\rho\,\pi\,x}{\ell} - N\sum_\rho c_\rho \left(\frac{\rho\,\pi}{\ell}\right)^2 \sin\frac{\rho\,\pi\,x}{\ell} \\ & - \sum_j \left\{ \frac{N}{\mu_j} \sum_\rho c_\rho \left(\frac{\rho\,\pi}{\ell}\right)^2 \sin\frac{\rho\,\pi\,x}{\ell} [1 - H(x - \alpha_j)] \right\} = -Nw_o'' - \sum_j \frac{N}{\mu_j} w_o''[1 - H(x - \alpha_j)] + \sum_i m_i \delta'(x - \alpha_j) \end{aligned}\right\} \qquad (4a)$$

where: $P_j = N / \mu_j$ (4b)

3. DETERMINATION OF CRITICAL LOADS

By setting $w_o = 0$ and $m_i = 0$, eq(4a) becomes:

$$\left.\begin{aligned} & E\,I\sum_{\rho} c_{\rho}\left(\frac{\rho\pi}{\ell}\right)^{4}\sin\frac{\rho\pi x}{\ell} - 2E\,I'\sum_{\rho} c_{\rho}\left(\frac{\rho\pi}{\ell}\right)^{3}\cos\frac{\rho\pi x}{\ell} - E\,I''\sum_{\rho} c_{\rho}\left(\frac{\rho\pi}{\ell}\right)^{2}\sin\frac{\rho\pi x}{\ell} \\ & - N\sum_{\rho} c_{\rho}\left(\frac{\rho\pi}{\ell}\right)^{2}\sin\frac{\rho\pi x}{\ell} - \sum_{j}\frac{N}{\mu_j}\sum_{\rho} c_{\rho}\left(\frac{\rho\pi}{\ell}\right)^{2}\sin\frac{\rho\pi x}{\ell}[1 - H(x-\alpha_j)] = 0 \end{aligned}\right\} \qquad (5)$$

Applying the Galerkin procedure in eq(5), and taking into account the orthogonality condition, we have:

$$c_1(A_{k1} - \alpha_{k1}N) + c_2(A_{k2} - \alpha_{k2}N) + \cdots\cdots + c_n(A_{kn} - \alpha_{kn}N) = 0 \quad \text{with} \quad k = 1 \text{ to } n \qquad (6a)$$

where:

$$\left.\begin{aligned} A_{k\rho} &= E\int_0^{\ell} I(x)\left(\frac{\rho\pi}{\ell}\right)^{4}\sin\frac{\rho\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx - 2E\int_0^{\ell} I'(x)\left(\frac{\rho\pi}{\ell}\right)^{3}\cos\frac{\rho\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx \\ &\quad - E\int_0^{\ell} I''(x)\left(\frac{\rho\pi}{\ell}\right)^{2}\sin\frac{\rho\pi x}{\ell}\sin\frac{k\pi x}{\ell}dx \\ \alpha_{k\rho} &= \frac{\rho^2\pi^2}{2\ell} + \mu_j\int_0^{\ell}\frac{\rho^2\pi^2}{\ell^2}\left(\sin\frac{\rho\pi x}{\ell}\right)^2[1 - H(x-\alpha_j)]dx \qquad \text{for} \quad \rho = k \\ \alpha_{k\rho} &= \frac{\rho^2\pi^2}{2\ell} + \mu_j\int_0^{\ell}\frac{\rho^2\pi^2}{\ell^2}\sin\frac{\rho\pi x}{\ell}\sin\frac{k\pi x}{\ell}[1 - H(x-\alpha_j)]dx \qquad \text{for} \quad \rho \neq k \end{aligned}\right\} \qquad (6b)$$

In order that the above linear homogeneous system eq(6a) without second member has non-trivial solutions, the determinant of the unknowns c_ρ must be equal to zero. The above-mentioned condition concludes to the following buckling equation:

$$\left\| A_{k\rho} - \alpha_{k\rho}N \right\| = 0 \qquad (7)$$

Equation (7) gives the spectrum of the critical buckling loads.

4. EQUILIBRIUM PATHS

The equilibrium paths of the problem can be determined by eq(4). Applying once again the Galerkin procedure in eq(4), and taking into account the orthogonality condition we get:

$$c_1(A_{k1} - \alpha_{k1}N) + c_2(A_{k2} - \alpha_{k2}N) + \cdots\cdots + c_n(A_{kn} - \alpha_{kn}N) = B_k \quad \text{with} \quad k = 1 \text{ to } n \qquad (8a)$$

where $A_{k\rho}$ and $\alpha_{k\rho}$ are given in eq(6b), and B_k is given by the following expression:

$$B_k = -N\int_0^{\ell} w_o''\sin\frac{k\pi x}{\ell}dx - \sum_j \mu_j N\int_0^{\alpha_j} w_o''\sin\frac{k\pi x}{\ell}dx - \sum_i m_i\frac{k\pi}{\ell}\cos\frac{k\pi\alpha_i}{\ell} \qquad (8b)$$

From the system of equations (8a), we obtain the constants $c_\rho(N)$, $(\rho = 1 \text{ to } n)$ and from eq(3) the corresponding equilibrium paths

$$w(x) = c_1(N)\sin\frac{\pi x}{\ell} + c_2(N)\sin\frac{2\pi x}{\ell} + \cdots\cdots + c_n(N)\sin\frac{n\pi x}{\ell} \qquad (9)$$

5. FAILURE CRITERION

It is obvious that in some cases (especially for short columns), the column will never reach the elastic buckling load N_{cr}. Thus, it is necessary to introduce a plasticity criterion, which may be expressed as follows:

$$\frac{N}{A(x)}+\frac{M(x)}{W(x)} \le \sigma_f \quad \text{or} \quad N+\frac{A(x)}{I(x)} z(x) M(x) \le \sigma_f A(x) \quad \text{or} \quad N \le \sigma_f A(x) - z(x)\lambda^2(x) M(x) \quad \text{or finally}$$

$$N \le \sigma_f A(x) - z(x)\,\lambda^2(x)\,\frac{M(x)}{\ell^2} \qquad (10)$$

where $z(x)$ is the distance of the external fiber of a column's cross-section at position x.

6. IMPERFECTIONS AND COLUMNS

6.1 Initial imperfections

Initial imperfections are usually caused by a bad stacking during the transportation of the steel members and, rather rarely, by constructional reasons. In any case, they usually have a parabolic type form, which can be expressed as follows:

$$w_o(x) = -4f\left(\frac{x}{\ell}\right)^2 + 4f\,\frac{x}{\ell} \qquad (11)$$

where f is the maximum deformation at $x = \ell/2$.

6.2. The columns

The columns may have one of the forms shown in Fig. 2. In Fig. 2a, we see a column, which its cross-section changes parabolically. In Fig. 2b a tapered beam is shown, while in Fig. 2c one can see a beam composed by two tapered pieces with different length. Finally, in Fig. 2d, a stepped beam is shown.

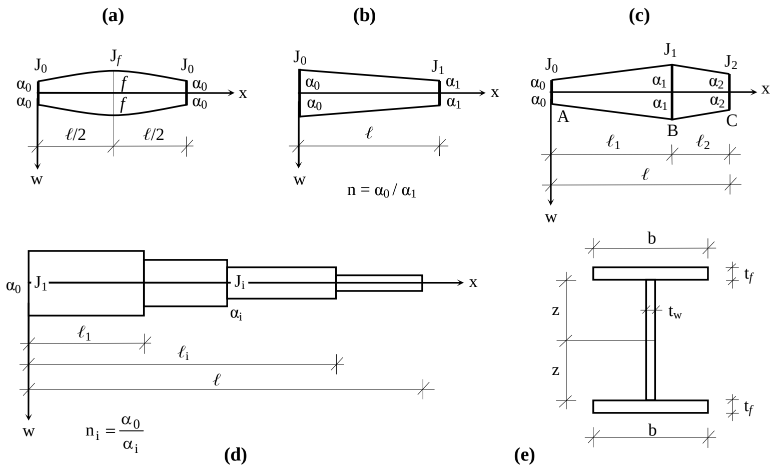

Fig. 2 Columns with different forms

In the cases of Fig. 2a,b,c, the cross-section, at any point x, has usually the same flanges ($b*t_f$) but a web with variable height z.
In any case, the following equation holds:

$$I(x) = \frac{4}{3}\cdot t_w \cdot z^3(x) + 2\,b\cdot t_f \cdot \left[z(x) + \frac{t_f}{2}\right]^2 \qquad (12)$$

6.2.1. The parabolic beam

If height z or moment of inertia I(x) change parabolically, it will be (see Fig. 2a and e):

$$\left.\begin{array}{ll} \text{z changes parabolically} & z(x) = 4(\alpha_o - f)\left(\frac{x}{\ell}\right)^2 - 4(\alpha_o - f)\cdot\frac{x}{\ell} + \alpha_o \\ \text{I changes parabolically} & I(x) = 4(I_o - I_f)\left(\frac{x}{\ell}\right)^2 - 4(I_o - I_f)\cdot\frac{x}{\ell} + I_o \end{array}\right\} \quad (13a,b)$$

6.2.2. The tapered beam

For the case shown in Fig. 2b, and assuming that $\alpha_1 = n\alpha_o$, we have:

$$z(x) = \alpha_o(n-1)\cdot\frac{x}{\ell} + \alpha_o \quad (14)$$

6.2.3. The beam consisting of two tapered pieces

For the case in Fig. 2c, and assuming that $\alpha_1 = n_1 \cdot \alpha_o$ and $\alpha_2 = n_2 \cdot \alpha_1$, we have:

$$\left.\begin{array}{ll} z(x) = \alpha_o(n_1 - 1)\cdot\frac{x}{\ell_1} + \alpha_o & \text{for} \quad 0 \le x \le \ell_1 \\ z(x) = n_1\alpha_o(n_2 - 1)\cdot\frac{x - \ell_1}{\ell_1} + n_1\alpha_o & \text{for} \quad \ell_1 \le x \le \ell \end{array}\right\} \quad (15a,b)$$

For these members we note that the existing discontinuity in $I'(x)$ at $x=\ell_1$, affects the results given by the use of some commercial mathematical programs. For the case of a beam composed by two pieces with lengths ℓ_1 and ℓ_2, and flanges AB and BC (Fig. 2c) that have the same inclination, we suggest the following approaching formula that removes the above-mentioned discontinuity:

$$z(x) = \alpha_o\left[\sqrt{\left[\frac{n-1}{\ell_1}\cdot(x-\ell_1)\right]^2 + \frac{1}{10\ 000}} - \left(n + \frac{1}{10\ 000}\right)\right] \quad \text{with} \quad n = \frac{\alpha_1}{\alpha_o} \quad (15c)$$

6.2.4. The stepped beam

In this case (Fig. 2d), we have:

$$I_i(x) = I_i = \text{constant} \qquad \text{for} \quad \sum_{\rho=1}^{i-1}\ell_\rho \le x \le \sum_{\rho=1}^{i}\ell_\rho \quad (16)$$

7. NUMERICAL RESULTS AND DISCUSSION

Table 1. Cross-sectional characteristics for common steel profiles

	IPE 200	IPE 400	IPE 600	HEB 200	HEB 400	HEB 600
2*z (m)	0.183	0.373	0.562	0.170	0.352	0.540
b (m)	0.100	0.180	0.220	0.200	0.300	0.300
t_w (m)	0.0056	0.0086	0.0120	0.0090	0.0135	0.0155
t_f (m)	0.0085	0.0135	0.0190	0.0150	0.0240	0.0300
A (m^2)	0.002735	0.008068	0.01510	0.00753	0.01915	0.02637
$I*10^{-4}$ (m^4)	0.213	2.559	10.604	0.5871	6.071	18.688

7.1. Critical loads

In the following figures, the plots of critical loads are drawn for beams with parabolic form made from IPE200 and HEB200 (Fig.3), with tapered form from IPE400 and HEB400 (Fig.4), and for a stepped beam (Fig.5), composed of three equal length pieces with $n_i=n$, where the piece with the smaller cross-section is IPE200. In Fig. 5a it is $\mu_1=0$ and $\mu_2=0$, while in Fig. 5b it is $\mu_1=5$ and $\mu_2=3$.

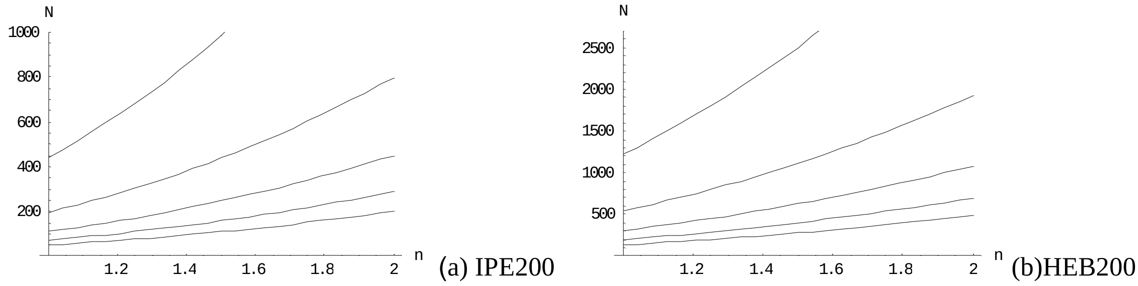

Fig. 3 Critical loads for parabolic beams of IPE200 and HEB 200 and lengths 10, 15, 20, 25 and 30m

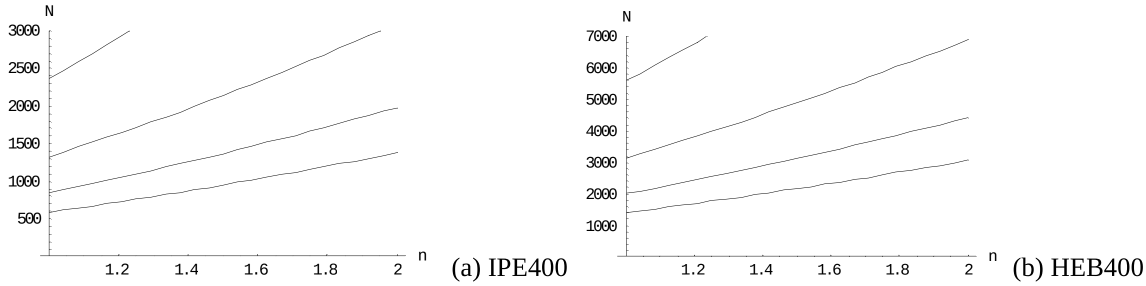

Fig. 4 Critical loads for tapered beams of IPE400 and HEB 400 and lengths 10, 15, 20, 25 and 30m

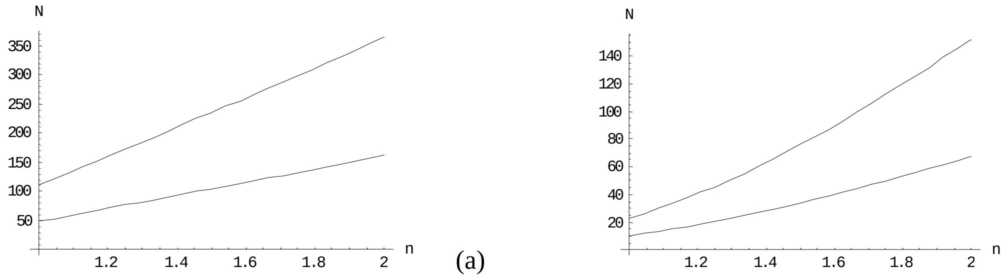

Fig. 5 Critical loads for a stepped beam of IPE200 of total length 20 or 30m, composed by 3 equal parts and loaded by a force N at ℓ, $P_1=\mu_1 N$ at ℓ_1 and $P_2=\mu_2 N$ at ℓ_2.

7.2. Equilibrium paths

In the plots shown in Fig. 6, the equilibrium paths versus N are shown for a beam with parabolic form and length 20m and n=1.25 (Fig.6a) and n=2 (Fig.6b) which is loaded by end-moments with magnitude 10, 20 and 30 kNm.

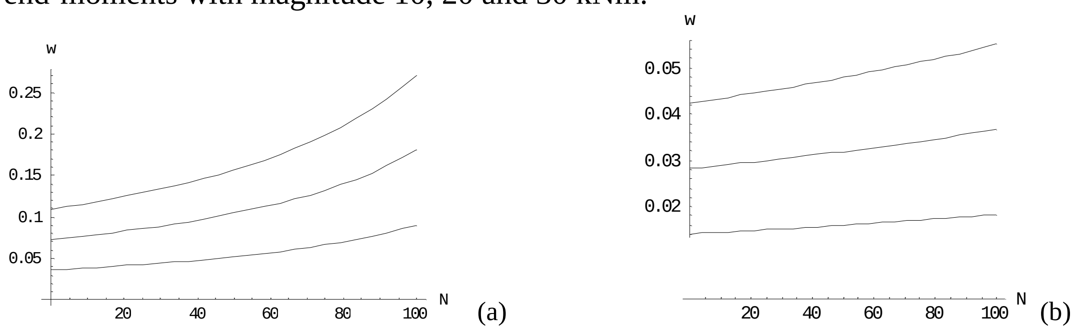

Fig. 6 Equilibrium paths of a parabolic beam of IPE200

7.3. Initial imperfections

In the plots of Fig. 7 the equilibrium paths versus N are shown for a tapered beam with length 20m, f=L/300, L/500, L/1000 and n=1.25 (Fig.7a) or n=2 (Fig.7b)

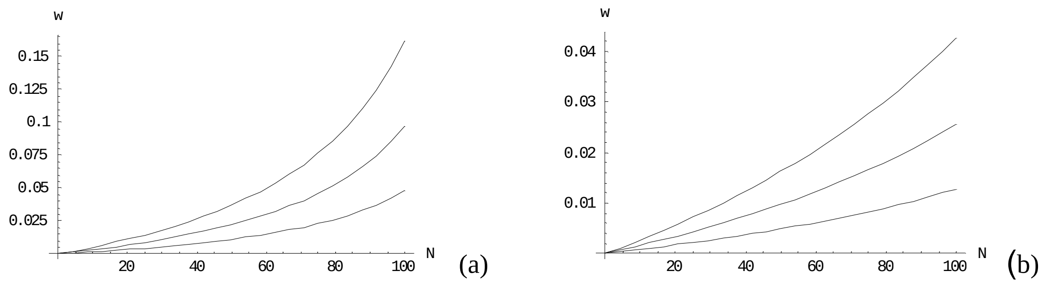

Fig. 7 Equilibrium paths of a tapered beam with initial imperfections

7.4. Eccentric loading

Finally, the plots in Fig. 8 show the equilibrium paths for the special case of a parabolic beam with length 10, 20, or 30m, with n=2, which is loaded by an axial force acting eccentrically at e=0.5m (Fig.8a) and e=1.0m (Fig.8b).

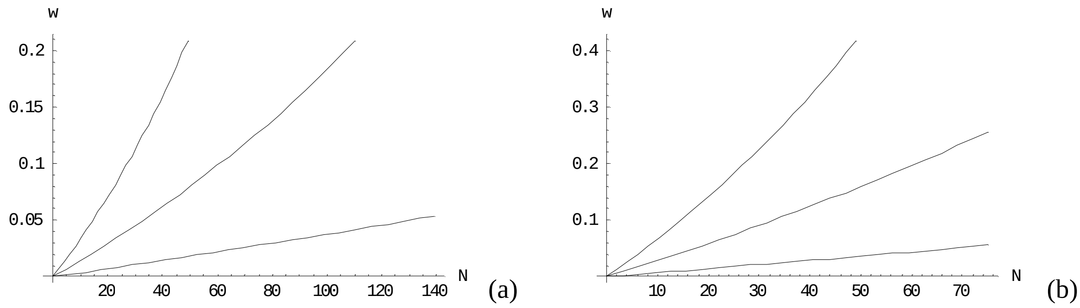

Fig. 8 Equilibrium paths of a parabolic beam loaded eccentrically

8. CONCLUSIONS

In this paper, the elastic stability of columns with any type of cross-sectional variation loaded by axial forces or bending moments acting at both its ends or at intermediate points is thoroughly studied.

The governing equation of the problem including all the above parameters is solved numerically following the Galerkin procedure and using the eigenshapes of the beam.

The law of the columns' cross-sectional variation may be of any type as well as the form of the initial imperfections. The accuracy of the method is proven excellent using only the three first eigenshapes. A plasticity criterion is applied in order to prevent material failure prior to buckling deformation.

The existence of bending moments or intermediate forces affects significantly not only the critical buckling load but also the load from the failure criterion.

The presented formulae are very simple in application to a personal computer employing a commercial program (such as MatLab, Mathematica, etc).

9. REFERENCES

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ΕΥΣΤΑΘΕΙΑ ΧΑΛΥΒΔΙΝΩΝ ΣΤΥΛΩΝ ΜΕ ΜΕΤΑΒΛΗΤΗ ΔΙΑΤΟΜΗ

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ΠΕΡΙΛΗΨΗ

Στην εργασία αυτή διερευνάται η ευστάθεια στύλων από δομικό χάλυβα με μεταβλητή διατομή που καταπονούνται σε αξονική θλίψη και κάμψη. Τούτο οφείλεται σε αξονικά φορτία και ροπές που δρουν είτε στα άκρα του μέλους ή σε οποιαδήποτε ενδιάμεση θέση. Μελετάται επίσης η επίδραση αρχικών γεωμετρικών ατελειών στην φέρουσα ικανότητα των στύλων αυτών καθώς και η μεταλυγισμική τους συμπεριφορά. Η ανάλυση που παρουσιάζεται εδώ βασίζεται στην επίλυση της διαφορικής εξίσωσης που διέπει το πρόβλημα με τη χρησιμοποίηση των ιδιομορφών της ράβδου. Τα αποτελέσματα παρουσιάζονται υπό τη μορφή διαγραμμάτων κρίσιμων φορτίων λυγισμού και δρόμων ισορροπίας και από αυτά εξάγονται χρήσιμα συμπεράσματα για το σχεδιασμό χαλύβδινων στύλων με μεταβλητή διατομή που είναι εντεταγμένοι σε πλαίσια..