IN-PLANE BUCKLING OF SHALLOW CIRCULAR STEEL ARCHES VIA CATASTROPHE THEORY

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1. ABSTRACT

Employing a two-mode harmonic approach, and based on the Theory of Catastrophes, the buckling response of a shallow steel circular arch is investigated. The outcome of this work indicates that the butterfly singularity governs the behavior of the system, a finding validated by numerical results.

2. INTRODUCTION

The analysis of shallow (flat) arches is more convenient from the viewpoint of energy [1], since the curvature term is negligible. The buckling of steel arches in particular, which may exhibit an essentially elastic behaviour even for large deformations, may be treated more comprehensively by exploring the nature of the total potential energy rather by solving the differential equilibrium equations under various loading conditions. A great number of publications have been reported in the literature regarding the buckling of shallow arches, starting from the 70's [2-5] and later on [6,7], but the approaches related to the analysis of the energy of the system are rather limited. Although more practical approaches have also been recently reported [8] a formulation based on Catastrophe Theory is marginal [9] and for convenience based on a sinusoidal initial arch shape. To this end the present work tackles the buckling response of steel circular shallow arches under point gravitational loading via Catastrophe theory, employing a two-mode harmonic approach. It was found that the butterfly singularity governs the system, a finding validated by numerical results. Further investigation is in order, so that possible higher-order instability phenomena may be explored.

3. PROBLEM STATEMENT

Let us consider an elastically supported shallow steel circular arch of uniform cross-section, acted upon by a gravitational point load P at an arbitrary point within its span, as shown in Figure 1.

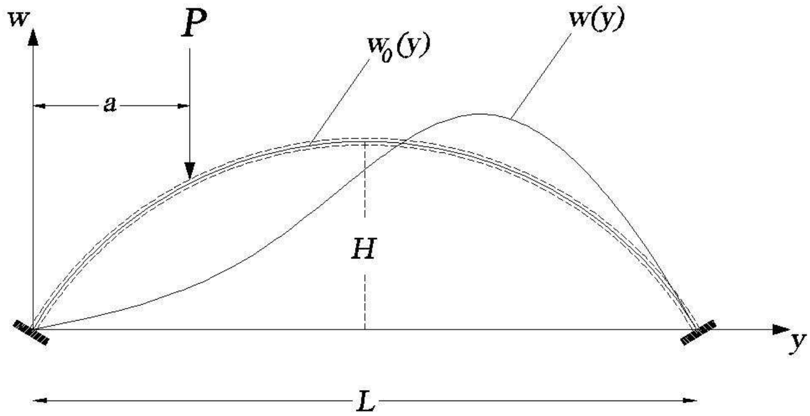


Fig. 1: Geometry and sign convention of a shallow circular arch

Denoting as $w_0(y)$ the initial unstressed circular configuration, given by

$$w_0(y) = \frac{4H}{L^2} y(L-y) \tag{1}$$

and by w(y) the deformed shape of the arch after the action of P, the truncated strain energy function (up to the fourth-order) of the system can be written as [7]:

$$U = \frac{1}{2} EI \int_0^L \begin{bmatrix} \left(w_0''^2 + w_0''^2 w_0'^2 + w_0''^2 w_0'^4\right) \\ -2w_0''\left(1 + \frac{1}{2}w_0'^2 + \frac{3}{8}w_0'^4\right)w'' + w''^2 \\ -w_0''\left(1 + \frac{1}{2}w_0'^2 + \frac{3}{8}w_0'^4\right)w''w'^2 + w''^2 w'^2 \end{bmatrix} dy \tag{2}$$

arranged in ascending powers of w and its derivatives, with the prime denoting differentiation with respect to y. We then employ a harmonic analysis and truncate the Fourier expansion of w(y), so that it contains only the leading two terms, i.e.

$$w(y) = Q_1 \sin\left(\pi \frac{y}{L}\right) + Q_2 \sin\left(2\pi \frac{y}{L}\right) \tag{3}$$

As indicated by Thompson and Hunt [7], this may seem as a somewhat crude approximation to a continuum with an infinite number of degrees of freedom, but is has been proven remarkably successful, since firstly it leads to closed-form solutions and may comprehensively capture the deformation of the arch, and secondly it has good agreement with experiments. Introducing the following dimensionless parameters

$$q_i = \frac{Q_i}{L} \;\; (i=1,2) \;,\; c = \frac{H}{L} \;,\; \alpha = \frac{a}{L} \;,\; y = \frac{x}{L},\; \bar{w}_0(x) = \frac{w_0(y)}{L} \;,\; \mathbf{w}(x) = \frac{w(y)}{L}$$
$$\lambda = \frac{P}{P_{cr}} \text{ where } P_{cr} = \frac{\pi^2 EI}{L^2} \tag{4}$$

it is valid that

$$\bar{w}_0(x) = 4cx(1-x) \tag{5}$$

$$\mathbf{w}(x)=q_1\sin(\pi x)+q_2\sin(2\pi x) \tag{6}$$

Evidently, the inextensibility of the shallow arch and the constraint of the immovable supports impose the fixed-length condition, so that q_1 and q_2 are not independent; this condition implies that the length of the arch before and after the action of the external load remains the same. Denoting as ℓ_w the length of the deformed structure, this is equal to [9]:

$$\ell_w=L+\frac{L}{4}\left[\left(\frac{\pi Q_1}{L}\right)^2+\left(\frac{2\pi Q_2}{L}\right)^2\right]-\frac{L}{8}\left[\frac{3}{8}\left(\frac{\pi Q_1}{L}\right)^4+\frac{3}{2}\left(\frac{\pi Q_1}{L}\right)^2\left(\frac{2\pi Q_2}{L}\right)^2+\frac{3}{8}\left(\frac{2\pi Q_2}{L}\right)^4\right] \tag{7}$$

and in dimensionless form, after a series expansion of $O(h^4)$:

$$\bar{\ell}_w=1+\pi^2\left(\frac{q_1^2}{4}+q_2^2\right)-\frac{3}{64}q_1^2\left(17q_1^2+16q_2^2\right) \tag{8}$$

The initial circular arc length equals to:

$$\bar{\ell}_{arc}=\frac{4c\sqrt{1+16c^2}+\mathrm{ArcSinh}(4c)}{8c}\cong 1+\frac{8c^2}{3}-\frac{32c^4}{5}+\mathrm{H.O.T.} \tag{9}$$

Hence, the aforementioned condition, i.e. $\bar{\ell}_w=\bar{\ell}_{arc}$, after cumbersome symbolic manipulations, leads to relations between the even powers of q_2 and parameter q_1, given by:

$$q_2^2=\frac{512c^2\left(5-12c^2\right)-48\left(5-40c^2+96c^4\right)\pi^2q_1^2+9\left(65+160c^2-384c^4\right)\pi^4q_1^4}{960\pi^2} \tag{10a}$$

$$q_2^4=\frac{1}{3600\pi^4}\begin{pmatrix}1024c^4\left(5-12c^2\right)^2+192\left(12c^2-5\right)\left(5-40c^2+96c^4\right)\pi^2q_1^2\\+9\left(25+12c^2\left(75+4c^2\left(55+96c^2\left(6c^2-5\right)\right)\right)\right)\pi^4q_1^4\end{pmatrix} \tag{10b}$$

On the other hand, the work of the external loading Ω is written as

$$\Omega=-P\left[w_0(a)-w(a)\right] \tag{11}$$

and in truncated dimensionless form

$$\bar{\Omega}=\frac{\Omega}{2P_{cr}L}\cong\frac{1}{12}\alpha\left(6\pi\left(q_1+2q_2\right)+24c\left(\alpha-1\right)-\pi^3\left(q_1+8q_2\right)\alpha^2\right)\lambda \tag{12}$$

The truncated total potential energy function of the system, normalized by $2P_{cr}L$, is thereafter calculated, by combining expressions (2), (4) – (6), (10a,b) and (12), leading to:

$$\bar{V}^{+,-}\left(q_1;\alpha,c,\lambda\right)=\frac{2V^{+,-}}{P_{cr}L}=\sum_{i=0}^{6}r_i^{+,-}q_1^i \tag{13}$$

where the superscript + corresponds to positive q_2 from expression (10a), and the – to the negative q_2 from the same expression. Hence, we came up to two potential functions of a single state variable – q_1 – which govern the system, both of a 6^{th} order polynomial type. The lengthy expressions corresponding to $r_i^{+,-}$ are not given herein for brevity; the nature of both potentials does not correspond to one of the seven elementary catastrophes [4,9], but rather to the extremely complicated star catastrophe, the geometrical representation, aspects and characteristics of which are far from common and rarely encountered in Structural Mechanics.

On the other hand, if the first order term q_2 (encountered only in $\bar{\Omega}$) is intentionally not substituted by its positive or negative value (in terms of q_1) resulting from (10a), the truncated dimensionless potential takes a form of:

$$\bar{V}\left(q_1, q_2; \alpha, c, \lambda\right) = \frac{2V}{P_{cr} L} = r_0(q_2, c, \lambda, \alpha) + \sum_{i=1}^{6} r_i(c, \lambda, \alpha) q_1^i \tag{14}$$

with coefficients r_i shown in Figure 2.

$$r_6 = \frac{3}{64}\left(-384 c^4 + 160 c^2 + 65\right)\pi^6$$

$$r_5 = \frac{c\left(384 c^4 - 160 c^2 - 65\right)\left(32\left(3\,262\,592 - 435\,600\,\pi^2 + 16\,875\,\pi^4\right)c^4 + 200\,\pi^2\left(-968 + 225\,\pi^2\right)c^2 + 5625\,\pi^4\right)}{9375\,\pi}$$

$$r_4 = \frac{1}{100}\left(55\,296\,c^8 - 46\,080\,c^6 + 4000\,c^2 + 425\right)\pi^4$$

$$r_3 = \frac{1}{84\,375\,\pi^3}\;32\left(4608\left(3\,262\,592 - 435\,600\,\pi^2 + 16\,875\,\pi^4\right)c^9 - 7680\left(815\,648 - 105\,270\,\pi^2 + 3375\,\pi^4\right)c^7 + 80\left(4\,907\,776 - 531\,600\,\pi^2 + 10\,125\,\pi^4\right)c^5 - 500\,\pi^2\left(1504 + 225\,\pi^2\right)c^3 + 28\,125\,\pi^4\,c\right)$$

$$r_2 = \frac{1}{150}\left(73\,728\,c^8 - 61\,440\,c^6 + 6080\,c^4 + 2800\,c^2 - 225\right)\pi^2$$

$$r_1 = -\frac{1}{337\,500\,\pi^5}\left(-786\,432\left(3\,262\,592 - 435\,600\,\pi^2 + 16\,875\,\pi^4\right)c^9 + 1\,310\,720\left(815\,648 - 105\,270\,\pi^2 + 3375\,\pi^4\right)c^7 + 512\,000\left(777\,600 - 101\,072\,\pi^2 + 2655\,\pi^4\right)c^5 + 28\,800\,000\,\pi^2\left(-24 + 5\,\pi^2\right)c^3 + 10\,800\,000\,\pi^4\,c + 28\,125\,\pi^6\,\alpha\left(\pi^2\alpha^2 - 6\right)\lambda\right)$$

$$r_0 = \frac{1}{225\,\pi^2}\left(73\,728\,\pi^2\,c^8 - 61\,440\left(-12 + \pi^2\right)c^6 + 1280\left(60 + \pi^2\right)c^4 + 4800\left(3 + \pi^2\right)c^2 + 450\,\pi^2\left(\alpha - 1\right)\alpha\,\lambda\,c + 75\,\pi^3\,q2\,\alpha\left(3 - 2\,\pi^2\alpha^2\right)\lambda\right)$$

Fig. 2: Coefficients of the truncated dimensionless potential given in eq. (14)

4. CATASTROPHE THEORY APPROACH

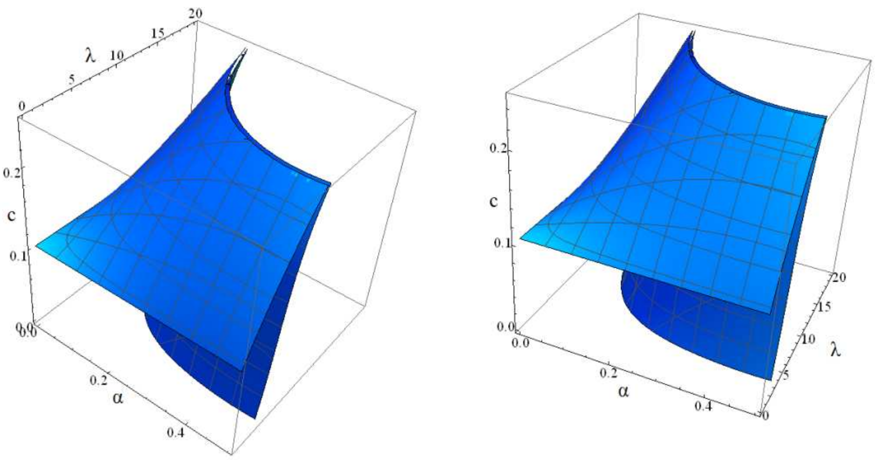
The potential energy functions given in eq.(13) are invariant under the sign change of q_2, as stated by Gilmore [9] and hence it is anticipated that the corresponding Bifurcational Sets differ infinitesimally. This set can be reconstructed by eliminating the state variable q_2 form the set of equations

$$V_1 = V_{11} = 0 \;\;,\;\; V_1 = \frac{d\bar{V}}{dq_1} \;\;,\;\; V_{11} = \frac{d^2\bar{V}}{dq_1^2} \tag{15}$$

and – since the product will obviously be a function of α, c and λ – the graphics involved can only by visualized via three dimensional contour plots.

On the other hand, for catastrophes of one generalized coordinate, only the butterfly is associated with a sixth order polynomial, but with missing the 5th order term and the constant term. Intuitively, one may resort to the second potential representation of eq. (14), set r_5=r_0=0 (thus ruling out the presence of q_2) and then reconstruct the corresponding Bifurcational Set. If this coincides with the Set evaluated from the star catastrophe potential, then the case of butterfly singularity will have been reached.

After performing the above reconstructions depicted in Figure 3, the obvious coincidence of these Bifurcational Sets leads to the conclusion, that the structure is governed by the butterfly catastrophe.



(a) (b)

Fig. 3: Coincident Bifurcational Sets for the star (a) and the butterfly (b) singularities for the shallow arch

For the butterfly catastrophe, it is in general valid that the control space is four-dimensional and that the bifurcation set is made of fold hypersurfaces (lines on a two-dimensional slice of control space) which meet on a surface of cusp-points, which themselves meet on lines of swallowtail points. Finally, these lines meet together at a butterfly point. At this point, the potential curve (germ) has only one minimum corresponding to the collapse of three minima and two maxima. Some of these features, taking into account, that the graphics in Figure 3 represent half the range of c, can be perceived in the above 3-D contours. It should also be noted, that the dual cusp and multiple folds, known to be exhibited by shallow arches [2,5,6] are also characteristic situations encountered in the butterfly singularity.

5. NUMERICAL RESULTS

In order to validate and verify that the system is governed by the butterfly singularity, the post-buckling deformation of the structure will be computed, based on the potential function given in expression (14). Equilibria are evaluated by solving the 1^{st} of equations given in (15), which deduces to the finding of the real roots of a quantic equation with respect to q_1. For all rational rates of the foregoing parameters, e.g. $c < 0.2$, $\alpha < 0.5$ and $\lambda < 20$ (half space due to symmetry) it was found that this equation has always either one or three real roots, and more specifically in the case of three roots, two of them correspond to the same post-buckling shape, related to either snap-through or bifurcational buckling, while the 3^{rd} root to the opposite response. These qualitative results are in strict conformity with the features of the butterfly catastrophe (exchange of stability via limit points and branching points) and will be demonstrated hereafter.

Our first application considers the case of a very shallow circular arch (c=0.10) acted upon by at its 1^{st} span quarter by a concentrated gravitational force (α=0.25). It was found that before a certain value of the loading, the system may be associated with either snap-through or bifurcational buckling, while beyond this value only a limit-point snapping behavior was encountered. Same results were produced for α=0.45 (much closer to the apex of the initial arch configuration) but for smaller values of the loading, as well expected. These results are illustrated in graphical form in the contents of Figures 4 and 5.

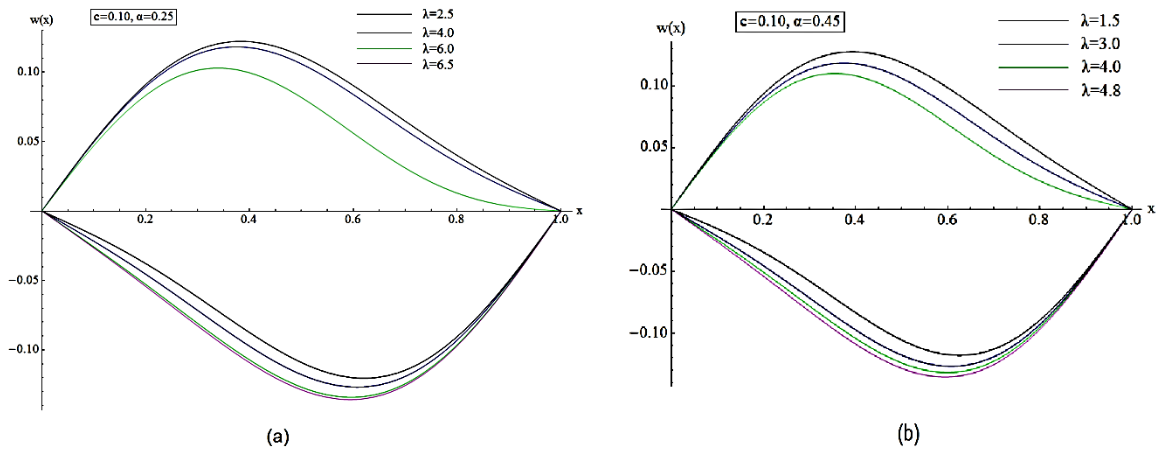


Fig. 3: Postbuckling deformation of a case related to a very shallow arch

On the contrary, totally different phenomena were assessed for near-shallow arches (e.g. c=0.20), with consecutive snapping and branching responses related to the 2nd buckling mode, as depicted in Figure 4. Less values of λ are required for the realization of the same phenomenon as the loading approaches the apex, i.e. the butterfly point of singularity.

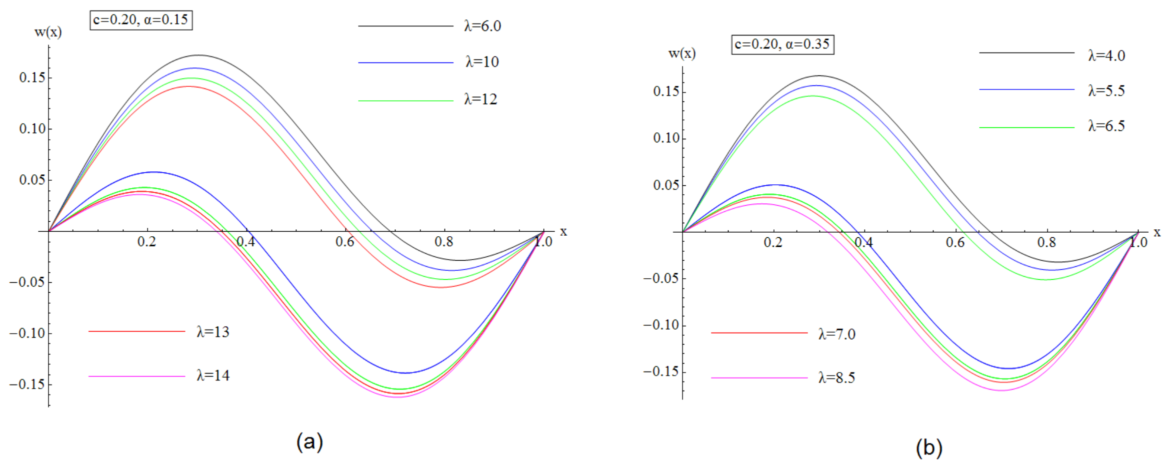


Fig. 4: Postbuckling deformation of a case related to a near-shallow arch

The above results are in very good agreement with existing ones in the literature [2-9] and strongly validate the finding regarding the butterfly singularity. Boundaries related to exchange of instability modes may be monitored by combining the analysis given above with the boundary conditions and the essential nonlinear differential equation of equilibrium. Since the proposed scheme is independent of the boundary conditions, provided that the supports of the arch are simple elastic and immovable, it can be applied for clamped and hinged ends.

6. CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

Using a two-mode Fourier harmonic analysis and on the basis of the Theory of Catastrophes it was found that a steel shallow arch – as far as in plane buckling under point gravitational loading is concerned – is governed by the butterfly singularity, implying folds, dual cusps and exchange of instability modes. Numerical results validate this finding, but it is suggested for future research that the analysis should be based on two state variables and a higher order Fourier discretization, so that more complicated postbuckling responses might be assessed.

7. REFERENCES

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ΕΝΤΟΣ ΕΠΙΠΕΔΟΥ ΛΥΓΙΣΜΟΣ ΧΑΜΗΛΩΝ ΧΑΛΥΒΔΙΝΩΝ ΚΥΚΛΙΚΩΝ ΤΟΞΩΝ ΜΕΣΩ ΤΗΣ ΘΕΩΡΙΑΣ ΚΑΤΑΣΤΡΟΦΩΝ

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ΠΕΡΙΛΗΨΗ

Εφαρμόζοντας μια αρμονική προσέγγιση δύο μορφών και με βάση τη Θεωρία των Καταστροφών, μελετήθηκε η συμπεριφορά σε λυγισμό ενός χαμηλού χαλύβδινου κυκλικού τόξου υπό συγκεντρωμένο φορτίο βαρύτητας και ελαστικές αμετάθετες στηρίξεις. Το αποτέλεσμα της έρευνας κατέδειξε ότι το σύστημα σχετίζεται με την ανωμαλία τύπου πεταλούδας, εύρημα το οποίο επαληθεύθηκε μέσω αριθμητικών αποτελεσμάτων.

Πιο συγκεκριμένα, το κολοβωμένο συνολικό δυναμικό του συστήματος έχει τη μορφή πολυωνύμου έκτου βαθμού με πλήρεις όλους τους όρους του, που όμως καταπίπτει σε πολυώνυμο ίδιου βαθμού αλλά χωρίς όρο 5^{ης} και μηδενικής τάξης (σταθερό), που αντιστοιχεί κατ' ευθείαν στην καταστροφή που προαναφέρθηκε. Αυτή σχετίζεται με πολλαπλές πτυχώσεις και διπλή αιχμή, που έχουν ήδη αναφερθεί στη σχετική βιβλιογραφία για τη απόκριση χαμηλών τόξων, με εναλλαγές της κρίσιμης συμπεριφοράς από λυγισμό οριακού σημείου σε διακλάδωση, όπως άλλωστε προέκυψε και από τις αριθμητικές εφαρμογές. Για περαιτέρω έρευνα προτείνεται η χρήση αρμονικής ανάλυσης υψηλότερης τάξης και υπολογισμός του συνολικού δυναμικού με δύο ενεργές συντεταγμένες, προκειμένου ανακαλυφθούν και άλλα, πιο σύνθετα φαινόμενα πιθανής μεταλυγισμικής συμπεριφοράς, ανάλογα με τις ισχύουσες κατά περίπτωση συνοριακές συνθήκες.