PLASTIC HINGE FORMATION IN FRAMES UNDER COMBINED STRESSES AND HARDENING BASED ON MATHEMATICAL PROGRAMMING

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1. ABSTRACT

In this work the elastoplastic analysis of frame structures with hardening behavior and axial force–bending moment interaction is examined in the framework of mathematical programming. The maximum load carrying capacity of the structure is determined by solving an optimization problem with linear equilibrium, compatibility and yield constraints together with a complementarity constraint that is of discrete rather than continuous nature. This is difficult to handle numerically and can be circumvented by several techniques, two of which are the penalty function and relaxation approach. These two methods are implemented for the analysis of steel frames for elastic-perfectly plastic and hardening behavior under pure bending and axial force-moment interaction. Numerical results are presented that reveal the inherent characteristics of the two methods in overall favoring the penalty method.

2. INTRODUCTION

Limit analysis has been extensively used for the elastoplastic analysis and design of structures. This aims at determining the collapse load and collapse mechanism that lead to a more efficient design following ultimate limit state design codes. Utilizing the potential of mathematical programming elastic-perfectly plastic behavior is fully explored. Furthermore, the work of Maier and his co-workers [1,2,3], has offered a whole new perspective at treating elastoplastic analysis problems obeying more general constitutive laws. Holonomic and non-holonomic elastoplastic analysis problems based on piecewise linear constitutive laws were formulated as quadratic programming problems or restricted basis linear programming problems or parametric linear complementarity problems. The combination of mathematical programming and limit analysis approach with multilinear constitutive laws has led to the formation of an optimization problem with equality constraints and a linear complementarity constraint that provides maximum load factor, stresses, displacements and strains at member ends simultaneously [4,5,6,7]. This problem is of discrete nature due to the presence of the complementarity constraint and is circumvented by means of various methods [8]. The aim of the present work is to compare two of the proposed methods for solving the aforementioned nonconvex optimization problem and to examine their robustness and efficiency in calculating the maximum collapse load for hardening behavior that accounts for the interaction of axial load and bending moment.

3. MATHEMATICAL MODEL AND GOVERNING RELATIONS

Plane frames are considered that consist of straight prismatic elements subjected only to nodal loading for reasons of simplicity. Frame displacements are assumed small enough so that the equilibrium equations refer to the initial undeformed configuration. It is also assumed that the structure consists of *nel* elements and has *nf* degrees of freedom, while *y* is the number of yield hyperplanes at each element end. The equilibrium of every element is described in terms of three independent stress resultants, namely axial force (s_1^i) and bending moment (s_2^i) at the "start" node *j* and bending moment (s_3^i) at the "end" node *k*, as shown in Fig. 1a. Equilibrium at each element is enforced and the six end forces are expressed in terms of the three independent stress resultants of the element. The structural equilibrium relationship [4,5] is then established as:

$$B \cdot s = a \cdot f \tag{1}$$

where B is the ($nf \times 3nel$) structural equilibrium matrix, s is a ($3nel \times 1$) vector for all primary stress resultants, a is a scalar load factor and f is a ($nf \times 1$) matrix of nodal loading. Compatibility which relates the member deformation q^i to the nodal displacements u^i (Fig.1b) follows a congruent relation for the whole structure and is given by the following linear relation:

$$q = B^T \cdot u \tag{2}$$

where q is the ($3nel \times 1$) strain vector and u is the ($nf \times 1$) nodal displacement vector.

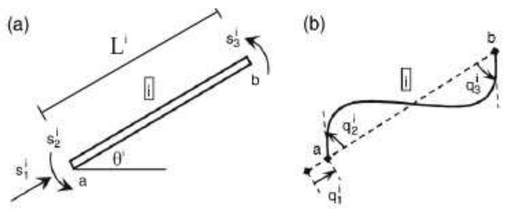

Fig. 1: Frame element i (a) the three independent stress resultants, (b) the corresponding generalized displacements.

The constitutive law decomposes the strain to an elastic and plastic part, as depicted in Fig.2. For the entire structure this is expressed by the relation:

$$q = e + p \tag{3}$$

where q is the total strain, e is the elastic and p the plastic strain.
The elastic branch is fully described by the relation:

$$s = S \cdot e \tag{4}$$

where S is the ($3nel \times 3nel$) stiffness matrix of the structure. The structural plastic deformations p are defined for holonomic assumption as follows:

$$p = N \cdot z \tag{5}$$

where N is the ($3nel \times 2ynel$) matrix of all unit normals to the yield hyperplanes and z is the ($2ynel \times 1$) vector of plastic multipliers. For the yield conditions a piecewise linearized locus is adopted for detection of plastic hinge formation at member ends accounting for axial-bending interaction [4,5]. This constitutes an inscribed polygon to the nonlinear yield condition and is an advantageous and safe approximation in limit analysis as it maintains the linearity of the constraint. For steel structures that are examined herein, a hexagonal piecewise linear yield locus is used as presented in Fig. 3 [4,7]. Positive and negative properties of the yield condition are identical and reduction of the pure bending capacity occurs for axial force greater than a fraction r_b (herein r_b considered as 0,15). Moreover, isotropic hardening behavior is considered, not complying though with Bauschinger effect. The set of hardening yield functions for the whole structure is collected in vector w ($2ynel \times 1$) as follows:

$$w = -N^T \cdot s + H \cdot z + r \geq 0 \tag{6}$$

where H is the ($2ynel \times 2ynel$) hardening matrix, z is the ($2ynel \times 1$) vector of plastic multipliers and r is the ($2ynel \times 1$) vector of yield limits. It is also noted that γ is the inclination angle defined in Fig. 3a , h is the tangent of the stress-strain diagram (Fig.3b), $\tau = 1 + 0{,}15 \cdot \tan\gamma$, $a_j = \dfrac{p_c}{\left|p_{cj}\right|}$ $j=1,\ldots,6$, where j is the corresponding number of the yield hyperplane, p_c is the arbitrarily assumed critical plastic strain and p_{cj} are the actual critical plastic strain values.

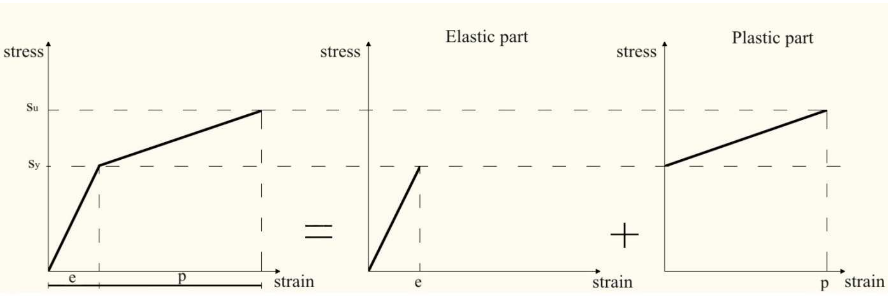

Fig. 2: Strain decomposition into elastic and plastic components.

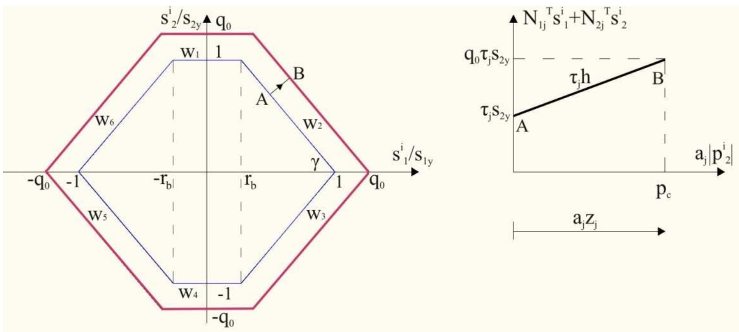

Fig. 3: (α) Yield and failure hyperplanes for axial force and bending interaction,
(b) Corresponding hardening behavior.

4. FORMULATION OF LIMIT ANALYSIS PROBLEM AND ALGORITHMS

Holonomic behavior is acceptable provided that no strain reversal (or "local" unloading) occurs in the structure during the loading process. For proportional loading and monotonically increasing loads, local unloading rarely occurs and, when it occurs, it seldom influences significantly the overall behavior, particularly for hardening structures [1]. The formulation of the holonomic problem consists of three basic notions, namely statics, kinematics and constitutive relations and is expressed within a Lagrangian small displacement regime by relations (1)-(6) together with the following complementarity condition:

$$w^T \cdot z = 0, \quad w \geq 0 \quad , \quad z \geq 0 \tag{7}$$

The latter prohibits simultaneous activation of plasticity and unloading. More specifically, the complementarity condition indicates that when the yield function w_j is activated ($w_j=0$), the corresponding plastic multiplier z_j should be greater than 0. Similarly, when the yield hyperplane j is inactive ($w_j>0$), the corresponding plastic multiplier $z_j=0$, namely no plastic flow occurs. Equations (1)-(7) can be simplified by retaining the variables s, u, z so that a Mixed Complementarity Problem (MCP) is formulated. This is equivalently converted into the following optimization problem the solution of which provides simultaneously the load multiplier a, stresses s, displacements u and plastic multipliers z [4]:

$$\left.\begin{array}{ll} \text{maximize} & a \\ \text{subject to} & B \cdot s - a \cdot f = 0 \\ & S^{-1} \cdot s - B^T \cdot u + N \cdot z = 0 \\ & w = -N^T \cdot s + H \cdot z + r \geq 0, \quad z \geq 0, \quad w^T \cdot z = 0 \end{array}\right\} \tag{8}$$

Mathematically this is a nonconvex optimization problem that is known as a Mathematical Programming with Equilibrium Constraints (MPEC) problem including the complementarity constraint that acts as a switch and is of discrete rather than continuous nature. This disjunctive constraint is difficult to handle numerically leading to numerical instabilities due to lack of convexity and smoothness. Despite all these inherent difficulties, the MPEC problem (8) can be solved by converting it into a standard, though still nonconvex, nonlinear programming (NLP) problem by suitably treating the complementarity condition. Several techniques have been proposed such as penalty function formulation, relaxation method, active set identification approach, sequential quadratic programming (SQP), interior point methods and others [8]. Herein, the penalty function approach (penalization) and relaxation approach (smoothing function) are investigated with respect to robustness and efficiency. The basic idea is to circumvent the complementarity constraint by a parametric reformulation, so that as the governing parameter increases (or decreases) the original complementarity condition is approached. According to penalization method, the complementarity term appropriately penalized is moved to the objective function [5] and the problem formulation is as follows:

$$\left.\begin{array}{ll} \text{maximize} & a - \rho \cdot w^T \cdot z \\ \text{subject to} & B \cdot s - a \cdot f = 0 \\ & S^{-1} \cdot s - B^T \cdot u + N \cdot z = 0 \\ & w = -N^T \cdot s + H \cdot z + r \geq 0, \quad z \geq 0 \end{array}\right\} \tag{9}$$

For the smoothing method, the objective function and the constraints are kept in their initial linear form and the complementarity constraint is replaced with a nonlinear function with an equivalent behavior [5]. Although there exists in literature a great number of such functions, herein the well known Fischer-Burmeister function is adopted given by:

$$\varphi_\varepsilon\left(z_j, w_j\right) = \frac{1}{2} \cdot \left(z_j + w_j - \sqrt{z_j^2 + w_j^2 + 2 \cdot \varepsilon^2}\right) \tag{10}$$

The parameter ε is iteratively decreased to satisfy the desirable complementarity tolerance. It is worth noting that the above two formulations are sensitive to the initial values of ρ and

ε and their subsequent increase and decrease respectively. Typical starting values of ρ and ε are between 0,1 and 1 with an update of $\rho=10\rho$ and $\varepsilon=\varepsilon/10$ after each NLP solution until an appropriate convergence tolerance is reached ($w^T z \leq 10^{-4}$).

5. PLASTIC ANALYSIS WITH MATHEMATICAL PROGRAMMING

The limit analysis problem described in relations (9) and (10) has been implemented in MATLAB code for the analysis of two steel frame structures (Fig. 4) [6,10] having material properties and geometrical characteristics presented in Table 1. The aim is to test the efficiency of the penalty function and relaxation approaches for the cases of elastic-perfectly plastic and hardening behavior under pure bending and axial force-moment interaction. In Table 2 all the results are presented. Firstly, it is noted that smoothing approach is more sensitive and falls short in efficiency and robustness as compared to penalization method. Although smoothing method gives a greater value of maximum load factor *a* for frame 2, the stress distribution to element ends seems to be limited and consequently fewer plastic hinges are formed which are more heavily damaged. It is also observed that frame 1 collapses at greater values of *a* for the case of pure bending than for the case of interaction when same material behavior is assumed, while frame 2 presents almost the same values of *a* for both cases. Moreover, it is noted that under the same conditions i.e. pure bending or combined stresses, incorporating hardening offers, as expected, greater values of maximum load factor *α* as compared to elastic-perfectly plastic behavior for both frames.

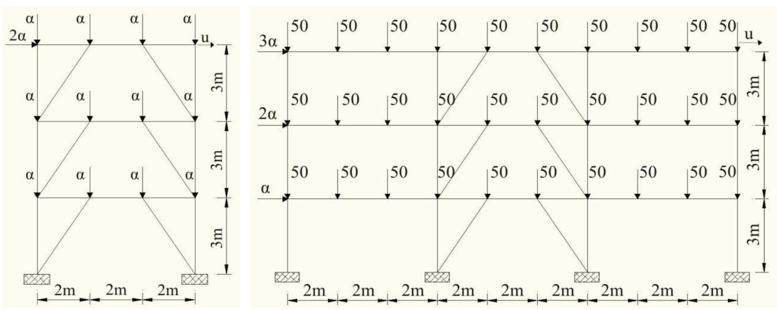

Fig. 4: Frames under examination (α) frame 1, (b) frame 2.

Material Properties & Geometrical Characteristics	FRAME 1			FRAME 2		
	Columns	Beams	Bracings	Columns	Beams	Bracings
Yield limit of axial force S_{1y}	3736,5 kN	669,3 kN	669,3 kN	3736,5 kN	669,3 kN	669,3 kN
Ultimate limit of axial force S_{1u}	5604,8 kN	803,2 kN	803,2 kN	5604,8 kN	803,2 kN	803,2 kN
Yield limit of bending moment S_{2y}	602,1 kNm	51,8 kNm	51,8 kNm	602,1 kNm	51,8 kNm	51,8 kNm
Ultimate limit of bending moment S_{2u}	903,1 kNm	62,2 kNm	62,2 kNm	903,1 kNm	62,2 kNm	62,2 kNm
Hardening *h*	200 kNm	40 kNm	40 kNm	200 kNm	20 kNm	20 kNm
Number of elements *nel*	21			45		
Number of nodes *nnodes*	14			34		
Number of degrees of freedom *nf*	36			90		
Modulus of elasticity *E*	$2\cdot10^8$ kN/m²			$2\cdot10^8$ kN/m²		

Table 1: Properties of frames under examination.

Maximum Load Factor a	Elastic-perfectly plastic (pure bending)	Elastic-perfectly plastic (interaction)	Penalization with hardening (pure bending)	Penalization with hardening (interaction)	Smoothing with hardening (interaction)
	FRAME 1				
	124,4 kN	122 kN	135,4 kN	131,4 kN	124,9 kN
	FRAME 2				
	82,8 kN	84,1 kN	91,8 kN	87,5 kN	112,9 kN

Table 2: Maximum load factor a for all limit analysis cases.

The aforementioned analysis cases are also conducted for increasingly applied load to determine the capacity curve of the structure. The maximum load factor a versus upper-storey horizontal displacement u is depicted in Fig.5. The conclusions generally coincide with those of limit analysis for single step loading and in addition, it is more evident that the assumption of elastic-perfectly plastic behavior leads to responses with great displacements and consequently great values of ductility. In Fig. 6, for hardening behavior consideration and for penalty function approach the plastic hinge disposition, the corresponding stress conditions of element start ends as well as the deformed shape of frames are displayed

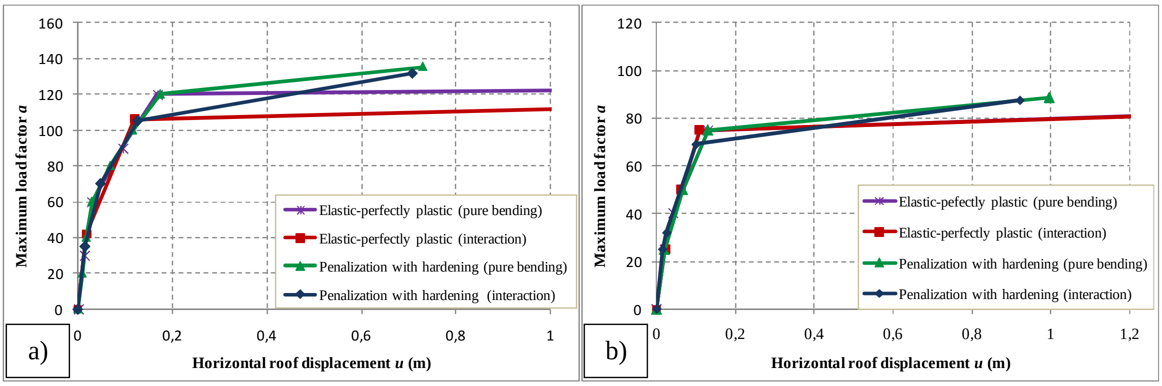

Fig. 5: Responses of (a) frame 1 and (b) frame 2.

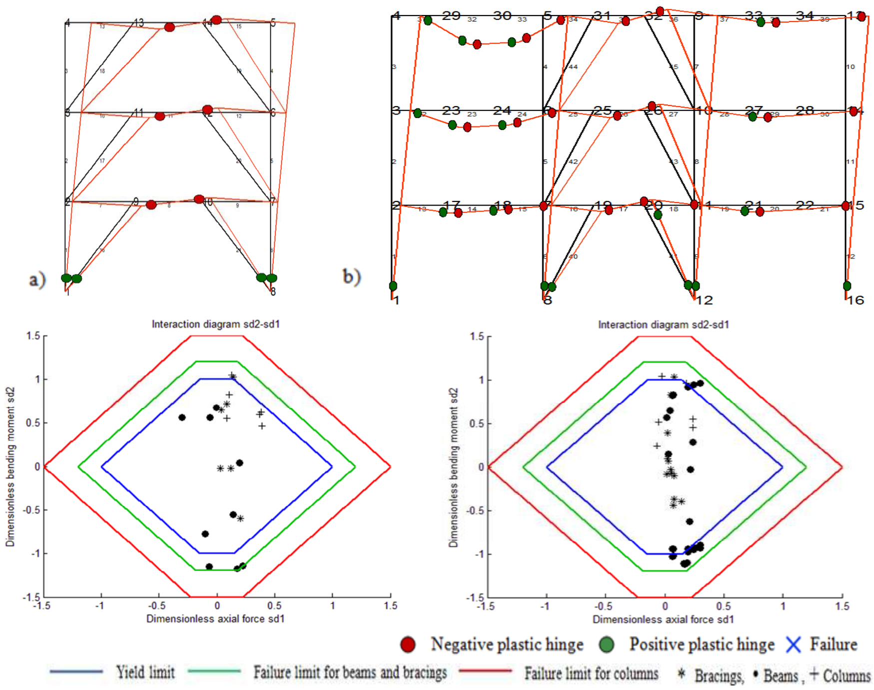

Fig. 6: Plastic hinge disposition and corresponding stress conditions of element start ends for penalization approach at (a) frame 1 and (b) frame 2.

6. CONCLUDING REMARKS

In this work, limit analysis of frame structures exhibiting hardening behavior is formulated as a mixed complementarity problem of mathematical programming. The determination of maximum load is established by solving an optimization problem with complementarity constraint. This is appropriately treated with either penalty function formulation or relaxation approach via a smoothing function. These methods are implemented for the analysis of two steel structures for the cases of elastic-perfectly plastic and hardening behavior under pure bending and axial force-bending moment interaction. The results proved that penalization method is more robust than smoothing which is more sensitive in setting the initial values, the bounds and the parameter that frequently needs to be altered in the course of the procedure to achieve accurate results.

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ΕΛΑΣΤΟΠΛΑΣΤΙΚΗ ΑΝΑΛΥΣΗ ΠΛΑΙΣΙΩΝ ΜΕ ΚΡΑΤΥΝΣΗ ΚΑΙ ΑΛΛΗΛΕΠΙΔΡΑΣΗ ΑΞΟΝΙΚΗΣ ΔΥΝΑΜΗΣ - ΡΟΠΗΣ ΜΕ ΧΡΗΣΗ ΜΕΘΟΔΩΝ ΜΑΘΗΜΑΤΙΚΟΥ ΠΡΟΓΡΑΜΜΑΤΙΣΜΟΥ

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ΠΕΡΙΛΗΨΗ

Η εργασία αυτή πραγματεύεται το θέμα της ελαστοπλαστικής ανάλυσης με χρήση μεθόδων μαθηματικού προγραμματισμού. Ο προσδιορισμός του φορτίου κατάρρευσης επιτυγχάνεται μέσα από την επίλυση ενός προβλήματος βελτιστοποίησης με γραμμικούς περιορισμούς ισορροπίας, συμβιβαστού παραμορφώσεων, διαρροής και έναν περιορισμό συμπληρωματικότητας, ο οποίος έχει διακριτό χαρακτήρα και δυσκολεύει την αριθμητική επίλυση, ενώ μετατρέπει το πρόβλημα σε μη κυρτό. Το εμπόδιο αυτό μπορεί να αρθεί με χρήση διάφορων μεθόδων, δύο εκ των οποίων είναι η εισαγωγή συνάρτησης ποινής ή συναρτήσεων εξομάλυνσης. Οι μέθοδοι αυτές εφαρμόζονται για την ανάλυση δύο μεταλλικών πλαισίων για διάφορες θεωρήσεις συμπεριφοράς υλικού και για διάφορους συνδυασμούς δράσεων. Τα αποτελέσματα δίνουν προβάδισμα στη μέθοδο ποινής, ενώ οι συναρτήσεις εξομάλυνσης προκύπτουν σημαντικά ευαίσθητες και ασταθείς.