A Levinson-type beam approach for double-T cross section beams in bending

Efthymios Koltsakis
Assistant Professor, Metal Structures Laboratory, Civil Eng. Dept.
Aristotle University of Thessaloniki, and
Researcher, Luleå Technical University, Sweden
Email: eftkol@ltu.se

Kimon Thomopoulos
Professor, Metal Structures Laboratory, Civil Eng. Dept.
Aristotle University of Thessaloniki,
Thessaloniki, Greece
Email: kthomop@civil.auth.gr

1. ABSTRACT

The present paper deals with the derivation of a Levinson - type solution for the bending of double-T cross section beam. It is generally accepted that the Euler-Bernoulli bending theory (EBT) looses accuracy for *L/h* ratios less than 10. The discrepancies become all the more significant in the case of alternating-curvature bending (herein referred to as anti-symmetric bending).
The existing bending theories that account for shear are the well known Timoshenko beam theory (TBT) and the more-recent Levinson beam theory (LBT), known also as the Reddy-Bickford theory, are all formulated for orthogonal cross sections. Numerical simulation shows significant differences in the behavior of double-T beams from the predictions of shear-aware existing bending theories derived with orthogonal cross sections in mind. The present paper attempts to address this problem by way of deriving and validating a bending theory that is particular for double-T beams.

2. INTRODUCTION

The problem of shear deformation of beams has been addressed by several researchers in the past. The best known attempt are those by Timoshenko [1] and Levinson [2], [3]. However both approaches were formulated around the concept of a beam with a thin orthogonal cross section. The case of H and I beams was not specifically addressed. These beams being of particular significance for the area of Steel Structures behave, as will be shown, somewhat differently. In the present paper an analytical approach that essentially is a modification of the Levinson beam theory (LBT) is presented and compared to Finite Element (FE) results.

3. FORMULATION

Levinson's original formulation [2] is classified by Reddy et al. [4] as a third order theory, due to the nature of the assumed displacement field

$$u(x,z) = z\psi(x) \ + \ z^3\phi(x) \tag{1}$$

Here the *x*-axis is taken along the length of the beam and *y, z* are the major and the minor axis of bending of the beam (Eurocode convention). The respective displacements are denoted by *u, v* and *w* and, in what follows, the origin of the reference system is taken at the middle of the beam which is assumed to be doubly symmetric. Levinson's key point is the remark that the *xz* component of the shear strain ϵ_{xz}has to vanish on the extreme fibers of the cross section as those are stress free. This gives rise to a relation between the two unknown functions $\phi$and $\psi$of eq.(1) as following

$$\phi(x) \ = \ -(4/3h^2)(\psi \ + \ \partial w/\partial x) \tag{2}$$

which is of crucial for the rest of the derivation. However, eq.(2) does not hold for the extreme fibers of the web of a double-T (2-T) beam as, the shear stress of the web has to balance-out the variation of the axial force of the flanges

$$\partial N_f/\partial x = Gt_w\epsilon_{xz}|_{z=\pm h/2} \tag{3}$$

The flange axial force for the assumed displacement field of eq.(1) can be written as

$$\frac{N_f}{Ebt_f} = \epsilon_{xx}|_{\pm h/2} = (h^3\phi'(x)/8 + h\psi'(x)/2) \tag{4}$$

Now, using eq.(1) in combination with the strain-displacement equations, eq.(3) yields

$$Ebt_f(h^3\phi''(x)/8 + h\psi''(x)/2) \ - \ Gt_w(\tfrac{3}{4}h^3\phi(x) + \psi(x) + w'(x)) = 0 \tag{5}$$

which relates $\phi$and $\psi$for in case of a 2-T beam in a way analogous to that of eq.(2) in the case of the rectangular cross section. The next step is to write down the equations of equilibrium of the beam making use of the expressions for the bending moment and the shear force that result from the assumed displacement field of eq.(2). First we write the expression for the strong axis bending moment in the form

$$M_y = Et_w\int_{-h/2}^{h/2} z\epsilon_{xx}dz \ + \ N_f \ h \tag{6}$$

where the contribution of the flanges appears as a separate term. Admittedly this expression captures only the membrane function of the flanges but numerical investigation showed the error to be less that 1%. Using eqs. (1), (4) we get

$$M_y \ = \ E(bh^4t_f/8 \ + \ h^5t_w/80)\ \phi'(x) + \ E(bh^2t_f/2 \ + \ h^3t_w/12)\ \psi'(x) \tag{7}$$

The shear force, obtained as the integral of τ_{xz}resulting from eq.(1), over the web, reads

$$Q_z = Gt_w \int_{-h/2}^{h/2} \epsilon_{xz} dz = Gt_w(\tfrac{1}{4}h^3\phi(x) + h\psi(x) + hw'(x)). \tag{8}$$

Now $M_y$and $Q_z$from eqs. (7),(8) will be substituted into the equilibrium of the beam

$$\partial Q_z/\partial x \ + \ p \ = \ 0 \ \text{ and } \ Q_z \ = \ - \ \partial M_y/\partial x \tag{9}$$

giving rise, for the case of two equal moments M_0at the ends, to

$$2M_0/L \ - \ m_1\psi''(x) \ - \ m_2\phi''(x) \ = \ 0 \tag{10}$$

and

$$Gt_w(\tfrac{1}{4} \ h^3\phi'(x) + h\psi'(x) + hw''(x)) \ = \ 0 \tag{11}$$

where

$$m_1 \ = \ E(bh^2t_f/2 \ + \ h^3t_w/12) \ , \ m_2 \ = \ E(bh^4t_f/8 \ + \ h^5t_w/80) \ . \tag{12}$$

The loading condition of antisymmetric bending is of particular interest here as, the Euler Bernoulli beam theory works with remarkable accuracy in the case of cylindrical bending. We now have a system of three equations, namely (5), (10) and (11) with three unknowns, that is the deflections $w(x)$and the functions $\phi(x)$and $\psi(x)$which fully define the stress situation in the beam.

4. ANALYTICAL SOLUTION

Due to the symmetry considerations, one can write down the following relations

$$w(x) = -w(-x), \ \psi(x) = \psi(-x), \ \phi(x) = \phi(-x) \tag{13}$$

The general solutions of the system are then calculated to be

$$\psi_0(x) = \frac{1}{2h^2Lm_1^2}\Bigg[- \frac{4Ebhm_2M_0t_f}{Gt_w} + m_1\Big[h^2\Big(-2M_0x^2 + 3Lm_2C_1 + 2Lm_1C_2\Big) + \\ + 4Lm_2(C_2 + C_4) - Lm_2\Big(3h^2C_1 + 4(C_2 + C_4)\Big)\cosh(\alpha x)\Big]\Bigg] \ , \tag{14}$$

$$\phi_0(x) = \frac{1}{2Gh^2Lm_1t_w}\Big[4bEhM_0t_f - GLm_1t_w\Big(h^2C_1 + 4(C_2 + C_4)\Big) + \\ + GLm_1t_w\Big(3h^2C_1 + 4(C_2 + C_4)\Big)\cosh(\alpha x) \tag{15}$$

$$w_0(x) = \frac{1}{48h^{5/2}Lm_1^2t_w}\Bigg[2\sqrt{h}x\ \Big[-12bEhM_0t_f(h^2m_1 - 4m_2) + \mathrm{G}\ \mathrm{m}_1t_w\Big(9h^4Lm_1C_1 - \\ - 48Lm_2(C_2 + C_4) + 4h^2\Big[2M_0x^2 - 9Lm_2C_1 + 3Lm_1C_2 + 9Lm_1C4\Big]\Big)\Big] +$$

$$+ 3L(h^2m_1-4m_2)^{3/2}\sqrt{Gm_1bEt_ft_w}\big(3h^2C_1+4(C_2+C_4)\big)\sinh(\alpha x)\Big] \tag{16}$$

where m_1, $m_2$were given in eq.(12) and

$$\alpha \;=\; \frac{2\sqrt{Ghm_1t_w}}{\sqrt{bE(h^2m_1-4m_2)t_f}} \tag{17}$$

The three constants of integration C_1, $C_2$and C_4remain to be determined. To this end, the following boundary conditions are considered: first the deflection is zero at the middle of the beam due to symmetry

$$w_0(L/2) \;=\; 0 \tag{18}$$

The shear is constant everywhere and equal to $2M_0/L$; using eq.(8) we write

$$Gt_w\big(\tfrac{1}{4}h^3\phi_0(x)+h\psi_0(x)+hw_0'(x)\big) \;=\; 2M_0/L \tag{19}$$

Finally, assuming the the end-moments $M_0$to be the result of pairs of concentrated forces applied at the flanges, one may consider the equilibrium along $x$of the upper part of the beam (imagine a full length longitudinal section along the $x$axis at some height $z$between $-h/2$and $h/2$) and write the condition

$$Gt_w\int_{-L/2}^{L/2}\epsilon_{xz}\;dx = 2M_0/h \tag{20}$$

The assumption that the end moments are the result of concentrated forces acting exclusively at the flanges is not inaccurate considering the majority of practical steel structures connections. Introducing the conditions (18)-(20) into (14)-(16) one obtains the values of the constants of integration

$$C_1 = -\frac{2M_0\mathrm{csch}(\alpha L/2)}{Gh^3Lt_w\sqrt{Em_1bt_f(h^2m_1-4m_2)}}\Big[\sqrt{Ght_w}\big(-2m_1+Ebh^2t_f\big)\;+$$
$$+\;2\;\sqrt{Em_1bt_f(h^2m_1-4m_2)}\sinh(\alpha L/2)\Big] \tag{21}$$

$$C_2 = \frac{M_0}{12Gm_1^{3/2}Lh^3t_w\sqrt{Ebt_f(h^2m_1-4m_2)}}$$
$$\Big[\sqrt{Em_1bt_f(h^2m_1-4m_2)}\big(24h^2m_1+48m_2-Gh^3L^2t_w\big)+$$
$$+\;24\;\mathrm{L}\;\mathrm{m}_2\sqrt{Ght_w}\big(-2m_1+Ebh^2t_f\big)\cosh(\alpha L/2)\Big] \tag{22}$$

$$C_4 = \frac{M_0\;\mathrm{csch}(\alpha L/2)}{12\;G\;m_1^{3/2}L\;h^3\;t_w\sqrt{Ebt_f(h^2m_1-4m_2)}}$$
$$\Big[\big(12h^2m_1-48m_2+Gh^3L^2t_w\big)\sqrt{Ebt_fm_1(h^2m_1-4m_2)}\sinh(\alpha L/2)+$$

$$+ 6L\sqrt{Ght_w}\left(h^2 m_1 - 4m_2\right)\left(-2m_1 + Ebh^2 t_f\right)\Bigg] \tag{23}$$

The evaluation of the constants of integration completes the procedure of the analytical solution. The hence obtained solution will be referred to as the modified Levinson beam theory (mLBT).

In Fig.1, the beam rotation taken as $(u(h/2) - u(-h/2))/h$of a simply supported HE500A beam subjected to two unit end-moments is plotted over it's length. Two *L/h* ratios, namely 4.5 and 20 are shown. The rotations are computed by means of classical EBT (blue), LBT (green) and the mLBT (red). All approaches converge for *L/h*=20 but, for *L/h*=4.5, some non-negligible discrepancies are evident. Note that EBT and LBT differ very little.

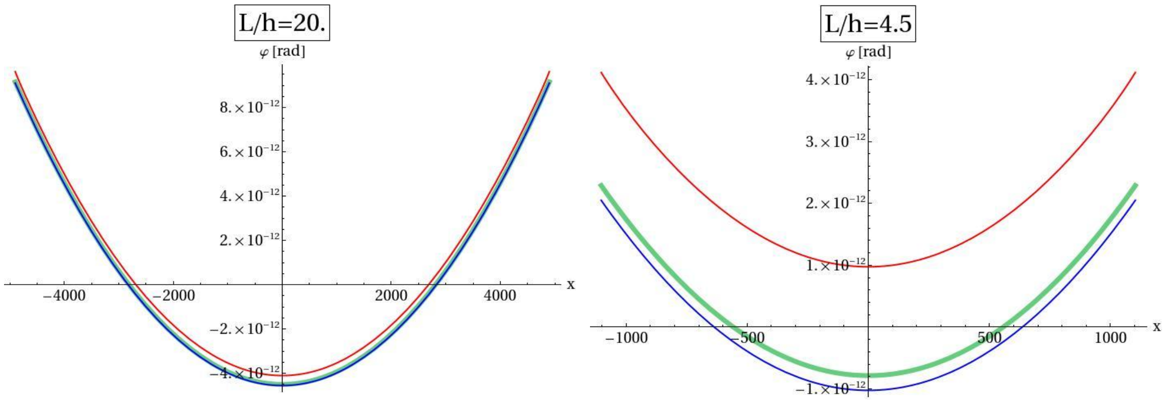

Fig.1 HE500A slopes by EBT (blue), LBT (green) and modified LBT (red)

5. NUMERICAL VERIFICATION

In what follows a brief account of the accuracy of the results of the mLBT will be given. The loading case considered is, as mentioned before, antisymmetric bending: simply-supported HE300A beam is subjected to two unit moments at it's ends. The beam length *L* is equal to *5h.* The horizontal axis of the graphs in Fig. 2 correspond to the *z* coordinate along the web height and the plotted quantity is the *x*-displacement sampled at four different positions indicated by ξ which is the normalized *x*-coordinate ($-0.5 \leqslant \xi \leqslant 0.5$).

The green dotted line shows the cross section *x*-displacement according to the Euler Bernoulli beam theory (EBT). The blue dotted line comes from the FE result. An 8-node quadrilateral reduce integration shell elements model with 14000 DOF was used. Finally the red line shows the result of the present mLBT. The differences between EBT and the FE/mLBT results is quite evident near the beam ends (ξ*=0.5,* ξ=0.479). However, it must be said that the discrepancies do not die-out as quickly for other cases. On the other hand, the mLBT always produces displacements close to the FE result. A total of 250 similar cases that were examined showed similar behavior. These included IPE, HEA and HEB beams from 100 to 550 with *L/h* ratios in the range of 5 to 25. A preliminary result not discussed in detail here due to space restrictions, is the increasing discrepancies between LBT and the present approach in proportion to the contribution of the flanges to the total moment of inertia of the cross section.

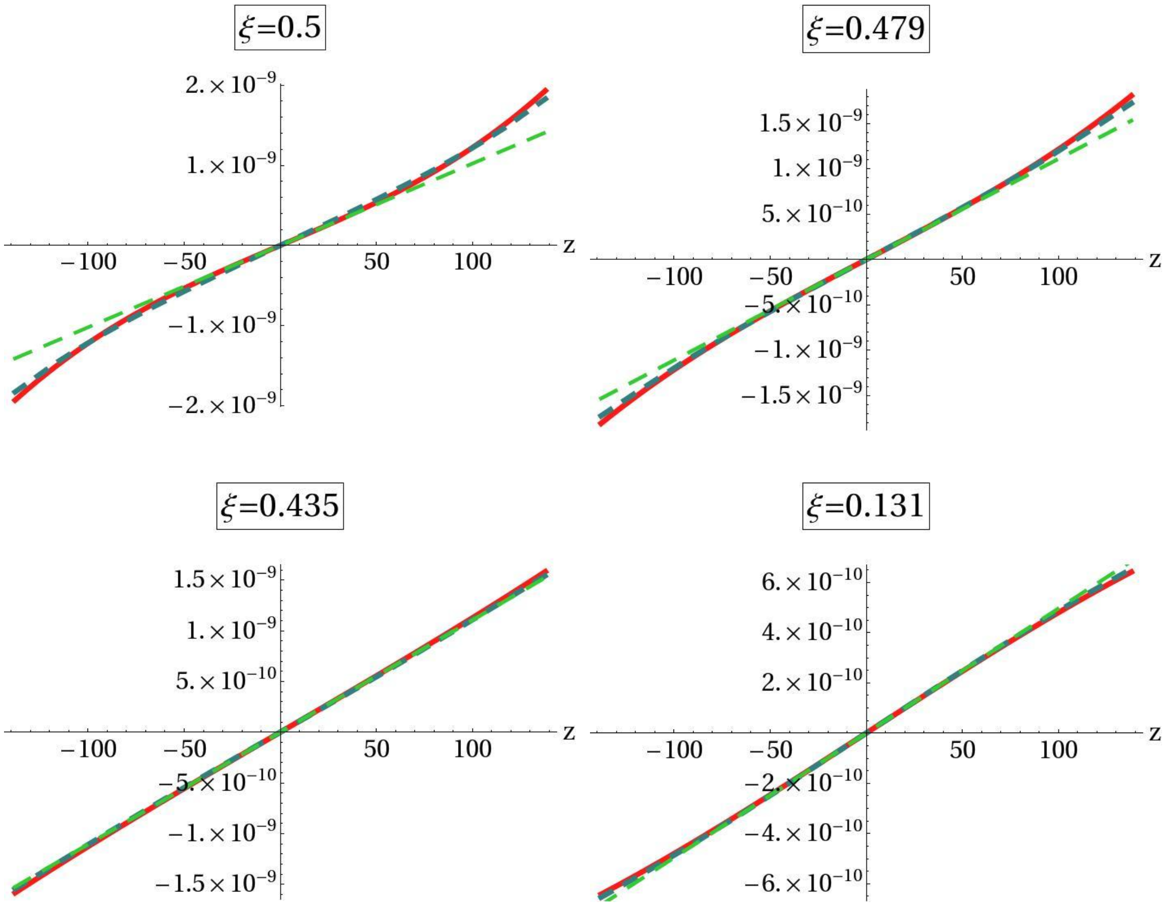

Fig. 2: x-displacements of a HE300A, L/h=5, by EBT (green), FE (blue) and mLBT (red)

6. CONCLUSIONS

A mathematical approach to the problem of anti-symmetric bending of double-T cross section beams was developed. A closed form solution was obtained and compared to the results of Finite Element models. Good convergence was shown between the theoretical and the numerical results. An other result is that beam theories formulated with the effects of shear in mind but, for orthogonal cross sections fail to accurately capture the problem of shear in the case of double-T beams, so widely used in the Steel Industry.

5. REFERENCES

[1] TIMOSHENKO S.P., WOINOWSKY-KRIEGER S., "Theory of Plates and Shells", McGraw-Hill, Singapore (1970).

[2] LEVINSON MARK, "A new rectangular beam theory", *Journal of Sound and Vibration,* Vol.74, 1981, pp.81-87.

[3] LEVINSON MARK, "Further results of a new beam theory", *Journal of Sound and Vibration,* Vol.77, 1981, pp.440-444.

[4] REDDY J.N., WANG C.M., LEE K.H., "Relationships between bending solutions of classical and shear deformation beam theories", *International Journal of Solids and Structures*, Vol.34, 1977, pp.3373-3384.

Μια προσέγγιση τύπου Levinson στο πρόβλημα της κάμψης δοκών διπλού-Τ

Ευθύμιος Κολτσάκης
Επίκουρος Καθηγητής, Εργαστήριο Μεταλλικών Κατασκευών
Τμήμα Πολιτικών Μηχανικών, Α.Π.Θ.
eftkol@ltu.se

Κίμων Θωμόπουλος
Καθηγητής, Εργαστήριο Μεταλλικών Κατασκευών
Τμήμα Πολιτικών Μηχανικών, Α.Π.Θ.
kthmop@civil.auth.gr

Η παρούσα εργασία πραγματεύεται το πρόβλημα της κάμψης υπό καθεστώς διάτμησης σε δοκούς διατομής διπλού Ταύ. Το πρόβλημα δεν είναι φυσικά καινούργιο και οι ευρύτερα γνωστές προσεγγίσεις είναι η δοκός Timoshenko και η δοκός Levinson η οποία είναι και γνωστή ως δοκός Reddy-Bickford. Οι θεωρίες αυτές έχουν αναπτυχθεί για ορθογωνικές διατομές. Αριθμητική διερεύνηση έδειξε αποκλίσεις των προβλέψεων από τα αποτελέσματα προσομοιώσεων με πεπερασμένα στοιχεία ιδίως της πλέον πρόσφατης προσέγγισης του Levinson (1981) που μπορεί να θεωρηθεί και ως αρτιότερη εκ των δύο. Περαιτέρω, μία προσεκτικότερη ανάγνωση της μόρφωσης του Levinson, δείχνει ότι το γεγονός της ανάληψης του κυρίου μέρους της καμπτικής ροπής από τα πέλματα των διατομών διπλού-Τ καθιστά μία από τις κύριες παραδοχές της προσέγγισης αυτής ανακριβή.

Στο κείμενο της παρούσας εργασίας παρουσιάζεται η τροποποιημένη μορφή της διατύπωσης της μόρφωσης του Levinson για τις διατομές διπλού-Τ και δίδεται λύση κλειστού τύπου για τις προκύπτουσες διαφορικές εξισώσεις για το πρόβλημα της αντισυμμετρικής κάμψης αμφιερίστου δοκού (δύο ίσες ροπές στα άκρα).

Η λύση συγκρίνεται με αριθμητικά αποτελέσματα πεπεραμένων στοιχείων όπου η δοκός προσομοιώνεται με επίπεδα στοιχεία κελύφους και βρίσκεται σε υψηλό επίπεδο συμφωνίας με αυτά.