CONVEX HULL FORMULATION FOR LIMIT STRUCTURAL ANALYSIS

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1. ABSTRACT

Limit analysis of framed structures based on static theorem is treated as a Linear Programming (LP) problem that maximizes the load factor under equilibrium and yield constraints. Yield condition is expressed herein in two different ways, i.e. i) as the intersection of half spaces and ii) as the convex hull/envelop of a fixed number of vertices. The two formulations differ in terms of number of variables and yield constraints and their computational efficiency is investigated for axial force-bending moment and axial-shear force- bending moment interaction. Numerical results of a 2D steel frame are presented that prove the computational advantages of convex hull formulation for both cases of interaction, demonstrating also the effect of combined stresses on the load carrying capacity.

2. INTRODUCTION

Limit analysis aims at determining directly the collapse load and collapse mechanism and is used for elastoplastic analysis and efficient design of structures. The Linear Programming (LP) formulation of the problem treats the ultimate load evaluation as an optimization problem and the two theorems of limit analysis, i.e. the static (lower bound) theorem and the kinematic (upper bound) theorem reflect the interrelation of primal-dual LP. Limit analysis with LP was initiated by Charnes and GreenBerg [1] for the ultimate state analysis of trusses. A variety of alternative mathematical programming procedures for limit analysis of discrete structures described by piecewise linear elastic-perfectly plastic constitutive laws were formulated and compared with respect to their computational merit by Maier et al. [2-6]. Most of this development is based on a piecewise linearization (PWL) of the convex yield condition that delimits the elastic domain as an intersection of half-spaces determined by a number of hyperplanes. The aim of this work is to express the yield condition as the convex hull of a number of vertices and compare in terms of computational efficiency the two different formulations for the yield condition, i.e. equations of yield hyperplanes (standard formulation) and convex hull formulation.

3. PROBLEM FORMULATION OF LIMIT ANALYSIS WITH LP

Plane frames are considered that consist of straight prismatic elements subjected only to nodal loading for reasons of simplicity. Frame displacements are assumed small enough so that the equilibrium equations refer to the initial undeformed configuration. It is also assumed that the structure consists of n_{el} elements and has n_f degrees of freedom, while d is the number of combined stresses (d-component interaction), h is the number of yield lines/hyperplanes and n_v is the number of vertices of the linearized yield surface.

Limit analysis based on the static theorem involves constraints concerning equilibrium and yielding. In this work, yield condition is established in two different ways: i) as a finite number of linear inequalities, which geometrically represents the intersection of a finite number of halfspaces and hyperplanes and ii) as convex hull of a fixed number of vertices.

A convex hull of a set of points C, is the space contained within the polyhedron that contains all points. Mathematically the convex hull or convex envelope of set C is the smallest convex set that contains C [7], and is expressed as ***conv*** C, containing all convex combinations of points in C:

$$\boldsymbol{conv}\, C = \{\theta_1 x_1 + \ldots + \theta_n x_n \mid x_i \in C, \quad \theta_i \geq 0,\ i = 1\ldots n, \quad \theta_1 + \ldots + \theta_n = 1\} \tag{1}$$

where θ_i are nonnegative coefficients and $x_1, \ldots, x_n$ are the points-vertices..

The aim of limit analysis is the determination of the ultimate load that a structure can sustain. The static approach involving equilibrium and yield conditions enforces the following LP formulation:

$$\begin{aligned} & \textit{maximize} \quad \alpha \\ & \textit{subject to} \quad \boldsymbol{B}\cdot\boldsymbol{s} - a\cdot\boldsymbol{f} = \boldsymbol{f}_d \qquad n_f \ \textit{constraints} \\ & \qquad\qquad\quad \boldsymbol{N}^T\cdot\boldsymbol{s} \leq \boldsymbol{r} \qquad\qquad 2h\,n_{el} \ \textit{constraints} \\ & \qquad\quad \boldsymbol{s}: \textit{unrestricted}, a \geq 0 \end{aligned} \tag{2}$$

where the decision variables are the stresses $\boldsymbol{s}$ and the load factor a. The first set of equality constraints constitutes the structural equilibrium relationship, where $\boldsymbol{B}$ is the ($n_f \times 3n_{el}$) structural equilibrium matrix, $\boldsymbol{s}$ is a ($3n_{el} \times 1$) vector for all primary stress resultants, a is a scalar load factor, $\boldsymbol{f}$ is a ($n_f \times 1$) matrix of nodal loading and $\boldsymbol{f}_d$ is the ($n_f \times 1$) fixed nodal load vector. The set of inequality constraints concerns the yield condition formulation, where $\boldsymbol{N}$ is the ($3n_{el} \times 2hn_{el}$) matrix of all scaled -with respect to yield capacities of stresses- normal vectors and $\boldsymbol{r}$ is the ($2hn_{el} \times 1$) vector that includes the yield limits of all yield lines. It is noted that, although the structure is statically indeterminate, compatibility relations are not considered, allowing the yield conditions to establish the collapse mechanism. For axial force-bending moment (NM) interaction h=8, while for axial-shear force-bending moment (NQM) interaction in sections $h=32$.

The formulation of limit analysis problem using convex hull consideration is given as:

$$\begin{aligned} & maximize \quad \alpha \\ & subject\ to \quad \boldsymbol{B}\cdot\boldsymbol{s} - a\cdot\boldsymbol{f} = \boldsymbol{f}_d \qquad n_f\ constraints \\ & \qquad \boldsymbol{T}\cdot\boldsymbol{s} - \boldsymbol{C}\cdot\boldsymbol{\theta} = 0 \qquad 2dn_{el}\ constraints \\ & \qquad \boldsymbol{I}_{eq}\cdot\boldsymbol{\theta} = 1 \qquad 2n_{el}\ constraints \\ & \boldsymbol{s}: unrestricted, \boldsymbol{\theta} \geq \boldsymbol{0}\ , a \geq 0 \end{aligned} \tag{3}$$

where the decision variables of the problem are the stresses $\boldsymbol{s}$, parameters $\boldsymbol{\theta}$ and the load factor a. The first set of equality constraints represents equilibrium, the second and third sets of constraints express yield condition using the concept of convex hull, where $\boldsymbol{T}$ is the $(2dn_{el}\times 3n_{el})$ matrix containing the yield capacities of the stresses, $\boldsymbol{C}$ is the $(2dn_{el}\times 2n_v n_{el})$ matrix containing the coordinates of the vertices of all yield hyperplanes for all elements, $\boldsymbol{\theta}$ is the $(2n_v n_{el}\times 1)$ vector including the non-negative coefficients θ_i for all vectors of the vertices n_v for all the elements, and $\boldsymbol{I_{eq}}$ is the $(2nel\times 2n_v n_{el})$ matrix that sums the corresponding θ_i at every element end. For axial force-bending moment (NM) interaction $d=2$ and $h=n_v=8$, while for axial-shear force-bending moment (NQM) interaction in sections $d=3$, $h=32$ and $n_v=18$.

It is noted that convex hull formulation expresses yield conditions with strict equality constraints, the number of which is independent of the number of the linear segments/planes of the yield surface. The number of variables, though, is increased as compared to the standard formulation since parameters θ_i are introduced. At this point, it is worth noting that generally an additional constraint requires more computational effort than an additional variable [8]. However, this is only indicative for the computational efficiency of convex hull formulation since the number of the new variables usually differs from the number of constraint reduction. Since for 3D interaction the number of vertices n_v is noticeably smaller than the number of planes h, convex hull formulation becomes considerably advantageous in terms of computational efficiency for expressing the yield condition.

4. NUMERICAL EXAMPLE

The optimization problems described in relations (2) and (3) are implemented in Matlab code for the analysis of frame steel structures with rigid-perfectly plastic behavior. The data are processed by *linprog* solver that is appropriate for linear programming problems. The aim is to compare the two formulations of yield condition and investigate the influence of axial force-bending moment interaction on the ultimate load. For this purpose, a steel plane frame is examined for the following cases:

- Case (a): Pure bending.
- Case (b): Axial force-bending moment interaction (NM interaction) with 1) standard formulation and 2) convex hull formulation.
- Case (c): Axial-shear force-bending moment interaction (NQM interaction) with 1) standard formulation and 2) convex hull formulation.

For case (a) the formulation of the problem is simplified since yield constraints consist of upper and lower bounds (side constraints) for the values of bending moments (no need of matrix $\boldsymbol{N}$). All analyses are conducted on a PC with a Core Duo Quad CPU and 4GB of RAM and the results of all cases are presented below. Notice that the analysis method follows the sign convention of matrix structural analysis, whereas final results are presented on the basis of engineering sign convention.

The example concerns the six-storey, four-bay plane frame, shown in Fig. 1, that is subjected to increasing lateral and fixed vertical loading. The frame is discretized into 72 elements, 56 nodes and 153 degrees of freedom. The steel grade is S235 with $E=2\times10^8$ kN/m^2. Sections with $A=197.5\times10^{-4}$ m^2, $I=86970\times10^{-8}$ m^4, $s_{1y}=4641.3$ kN, $v_y=1013.24$ kN, $s_{2y}=928.02$ kNm, $s_{3y}=928.02$ kNm and sections with $A=84.46\times10^{-4}$ m^2, $I=23130\times10^{-8}$ m^4, $s_{1y}=1984$ kN, $v_y=579.22$ kN, $s_{2y}=307.15$ kNm, $s_{3y}=307.15$ kNm are employed for all columns and beams respectively. Analysis results of all cases are presented in Table 1.

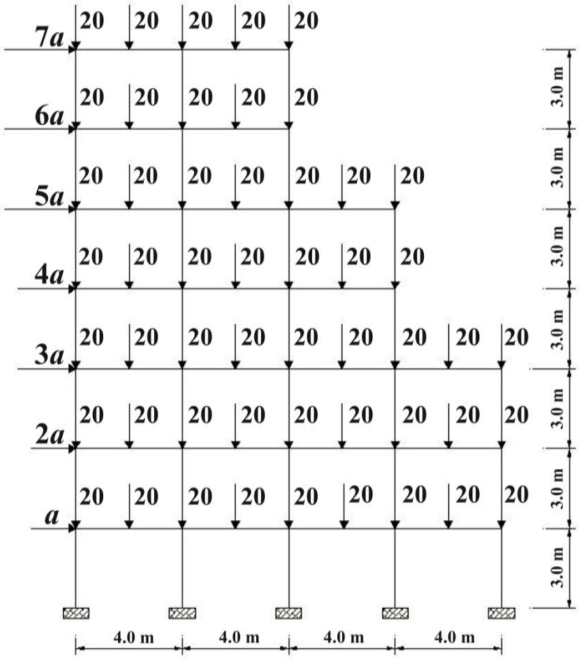

Fig. 1: Six-storey, four-bay plane steel frame.

Cases	Pure Bending (a)	NM (b$_1$)	NM Convex Hull (b$_2$)	NQM (c$_1$)	NQM Convex Hull (c$_2$)
maximum load factor ***a*** (kN)	**43.26**	**40.92**	**40.92**	**36.29**	**36.29**
number of plastic hinges	**49**	**51**	**51**	**52**	**52**
computational time (s)	**0.414**	**0.935**	**0.911**	**15.586**	**1.233**
number of variables n_{var}	**220**	**220**	**1388**	**220**	**2848**
number of equality constraints n_{eq}	**153**	**153**	**591**	**153**	**737**
number of inequality constraints n_{inq}	—	**1168**	—	**4672**	—

Table 1: Analysis results of all cases.

The axial-shear force-bending moment interaction corresponds to the smallest value of the maximum load factor and pure bending consideration to an unsafe greater value, as expected. Fewer plastic hinges are formed for case (a) that reach their yield limit in terms of bending moment. Analysis results of standard and convex hull formulation are identical for both NM and NQM interaction respectively. The computational efficiency of convex hull formulation is more pronounced for NQM interaction, since convergence is achieved 12.64 times faster. The plastic hinge patterns (number and location) and the corresponding interaction diagrams for cases (b) and (c) are shown in Fig. 2 and Fig. 3 respectively. The effect of combined stresses is evident at the yielded column cross sections (Fig. 2a and 3a) that under pure bending consideration remain elastic. The frame is mainly stressed due to bending moment (the dispersion of stress points is greater along the bending moment axis) for all cases, while the effect of shear force for some beam and column cross sections is more intense than that of axial force, as shown in Fig. 4b.

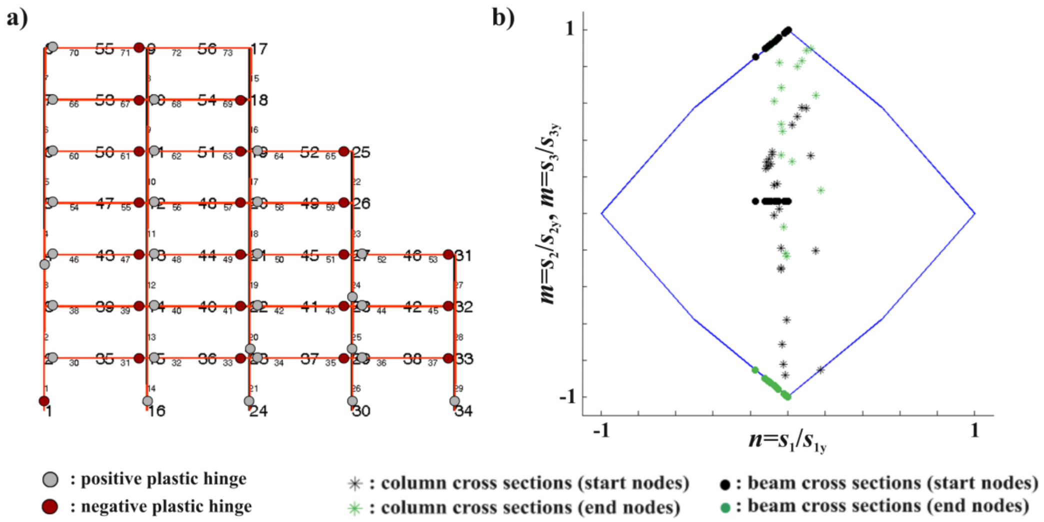

Fig. 2: a) *Plastic hinge pattern and b) interaction diagram for NM interaction.*

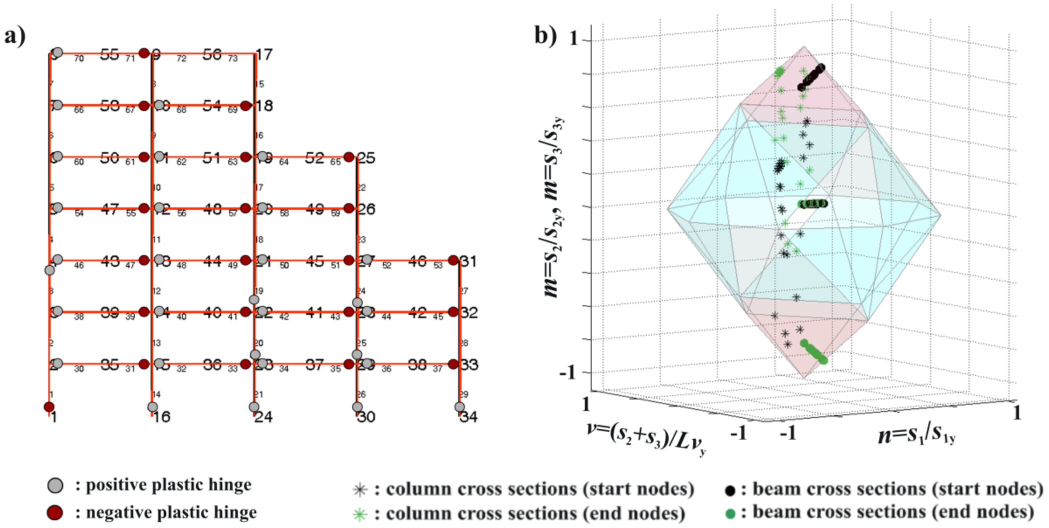

Fig. 3: a) *Plastic hinge pattern and b) interaction diagram for NQM interaction.*

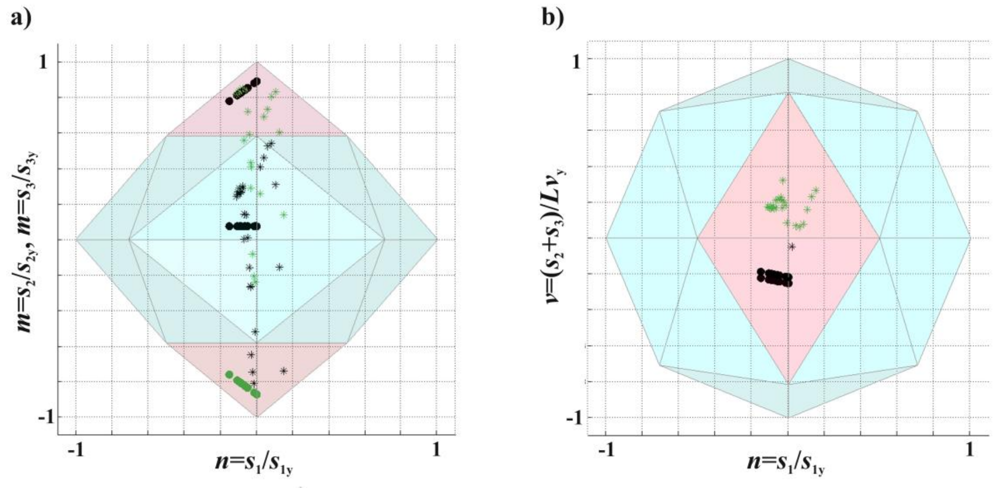

Fig. 4: Plan views of NQM interaction diagram.

5. CONCLUDING REMARKS

Limit analysis is treated herein in the framework of mathematical programming. The ultimate state of a structure and its maximum load carrying capacity is determined by solving a linear programming problem that aims at maximizing the load factor a subjected to constraints that enforce equilibrium and yielding (static theorem). In this work, two formulations of yield condition i.e. the yield polyhedron are examined, i) as the intersection of a finite number of halfspaces and hyperplanes (standard formulation) and ii) as a convex hull. The latter is equivalent to a set of equality constraints, the number of which is independent of the number of the yield hyperplanes. The price, though, is in the number of variables which is increased compared to the standard formulation, since nonnegative coefficients θ are introduced. The two formulations differ in terms of number of variables and constraints and are compared in terms of computational efficiency for axial force-bending moment (NM) and axial-shear force-bending moment interaction (NQM). Numerical examples are presented proving that convex hull formulation requires significantly less computing time compared to the standard one. More specifically, for NQM interaction the computational efficiency of convex hull formulation corresponds to a more than 13 times faster solution, compared to the standard one. This is due to the reduced number of constraints, since it is independent of the linearization of the yield surface, contrary to the standard formulation. Moreover, the increased number of variables for convex hull formulation is also associated with the number of vertices, which is noticeably smaller compared to the number of planes for the case of 3D interaction. Thus, convex hull formulation expresses advantageously multi-component interaction of the yield condition enabling fine approximations of the nonlinear yield surface. Furthermore, the effect of axial force-bending moment and axial-shear force-bending moment interaction on the ultimate load and state of a structure is demonstrated. The combined stresses generally correspond to reduced maximum load factors and to collapse mechanisms with more plastic hinges compared to pure bending consideration. Moreover, the effect of shear force on the ultimate load carrying capacity is remarkable, even for frames that are stressed mainly due to bending moment.

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REFERENCES

[1] CHARNES A., GREENBERG H.J. “Plastic collapse and linear programming”, *Bulletin of the American Mathematics Society,* Vol. 57, 1951, pp. 480.

[2] DONATO O.D., MAIER G. “Mathematical programming methods for the inelastic analysis of reinforced concrete frames allowing for limited rotation capacity”, *International Journal for Numerical Methods in Engineering*, Vol.4, 1972, pp. 307–29.

[3] MAIER G, GIACOMINI S, PATERLINI F. “Combined elastoplastic and limit analysis via restricted basis linear programming”, *Computer Methods in Applied Mechanics and Engineering*, Vol.19, 1979, pp. 21–48.

[4] MAIER G. “A matrix structural theory of piecewise linear elastoplasticity with interacting yield planes”, *Meccanica*, Vol.5, 1970, pp. 54-66.

[5] MAIER G, GRIERSON DE, BEST MJ. Mathematical programming methods for deformation analysis at plastic collapse. Comp Struct 1977; 7:599-612.

[6] COCCHETTI G., MAIER G. “Elastic–plastic and limit-state analyses of frames with softening plastic–hinge models by mathematical programming”, *International Journal of Solids and Structures*, Vol.40, 2003, pp. 7219–44.

[7] BOYD S., VANDENBERGHE L. “Convex Optimization”, seventh ed., Cambridge University Press, 2009.

[8] RAO S. “Engineering Optimization: theory and practice”, fourth ed., John Wiley & Sons, Inc., Hoboken, New Jersey, 2009.

ΟΡΙΑΚΗ ΑΝΑΛΥΣΗ ΚΑΤΑΣΚΕΥΩΝ ΜΕ ΘΕΩΡΗΣΗ ΚΥΡΤΟΥ ΠΟΛΥΕΔΡΟΥ (CONVEX HULL)

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ΠΕΡΙΛΗΨΗ

Στην παρούσα εργασία, η οριακή ανάλυση πλαισιακών φορέων, βασισμένη στο στατικό θεώρημα, αντιμετωπίζεται ως ένα πρόβλημα γραμμικού προγραμματισμού, το οποίο μεγιστοποιεί το φορτικό συντελεστή υπό περιορισμούς ισορροπίας και διαρροής. Η συνθήκη διαρροής εκφράζεται με δύο διαφορετικούς τρόπους: 1) ως τομή ημιχώρων, 2) ως κυρτό πολύεδρο (convex hull/envelop), που περιβάλλει συγκεκριμένες κορυφές. Οι δύο διατυπώσεις διαφέρουν ως προς τον αριθμό των μεταβλητών και των περιορισμών και διερευνάται η αποτελεσματικότητά τους για αλληλεπίδραση αξονικής δύναμης-καμπτικής ροπής και αξονικής-τέμνουσας δύναμης-καμπτικής ροπής. Παρουσιάζονται τα αποτελέσματα της ανάλυσης ενός επίπεδου, μεταλλικού πλαισίου, τα οποία αποδεικνύουν τα υπολογιστικά πλεονεκτήματα της διατύπωσης της περιβάλλουσας πολυγωνικής γραμμής και για τις δυο περιπτώσεις αλληλεπίδρασης. Επίσης, αναδεικνύεται η επίδραση των συνδυασμένων δράσεων στο οριακό φορτίο μιας κατασκευής.