DESIGN OF INTERMEDIATE RING STIFFENERS FOR COLUMN-SUPPORTED CYLINDRICAL STEEL SHELLS

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1. ABSTRACT

A combination of a ring beam and an intermediate ring stiffener can be used for large silos to redistribute the stresses from the local support into uniform stresses in the shell. This paper explores the design requirements for intermediate ring stiffeners placed at or below the ideal location. Pursuant to this goal, the cylindrical shell below the intermediate ring stiffener is analyzed using the membrane theory of shells and the reactions produced by the stiffener on the shell are identified. Furthermore, the displacements imposed by the shell on the intermediate ring stiffener are obtained. These force and displacement boundary conditions are then applied to the intermediate ring stiffener to derive closed form expressions for the variation of the stress resultants around the circumference to obtain a strength design criterion for the stiffener. These analytical studies are then compared with complementary finite element analyses to verify closed-form design equations for ring stiffeners.

2. INTRODUCTION

Discrete supports in cylindrical metal silos introduce local forces into the shell and produce a circumferential non-uniformity in the axial membrane stresses in the silo, which must be taken into account in assessing the stability of the shell. A combination of a ring beam and an intermediate ring stiffener can be used for large silos to redistribute the stresses from the local support into uniform stresses in the shell as shown in *Fig. 1*.
Greiner [1] and Öry and Reimerdes [2] showed that an intermediate ring stiffener can be very effective in reducing the circumferential non-uniformity of axial stresses in the shell.

Studies conducted by these researchers identified the variation of the axial membrane stress distributions up the height of the shell. It was shown that an intermediate ring stiffener can achieve a dramatic decrease in the peak axial membrane stress, producing a more uniform stress state above the intermediate ring. Recently Topkaya and Rotter [3] showed that there is an ideal location for an intermediate ring stiffener, such that the axial membrane stress above this ring is circumferentially completely uniform. The ideal location is identified by the height H_I above the ring beam, defined as the vertical distance between the top of the ring beam and the centre of the intermediate ring stiffener as shown in *Fig. 1*. This was determined analytically and is expressed (for the case where $\nu = 0.3$) in terms of basic geometric variables as follows:

$$H_I = \sqrt{12(1+\nu)}\frac{r}{n} = 3.95\frac{r}{n} \approx \frac{4r}{n} \quad (1)$$

where n = number of uniformly spaced column supports; r = middle surface radius; and ν = Poisson's ratio.

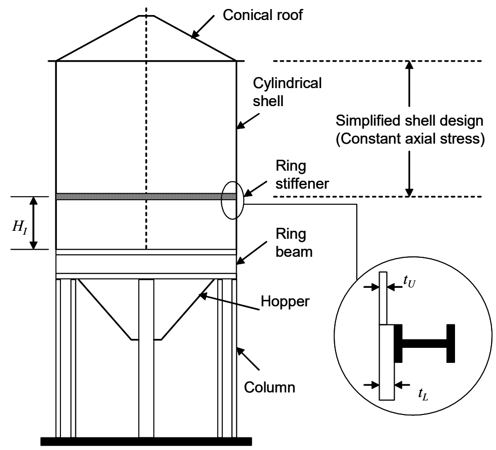

Fig. 1: Typical circular planform silo

In cases where a shell with large radius rests on a few supports, the ideal location can be quite high and the option of placing the intermediate ring stiffener below the ideal height may provide a viable solution [4].

This paper explores the strength requirements for intermediate stiffeners placed at ideal location or below this location, where the force transfer and displacement boundary conditions differ from those for a ring at the ideal location. A general shell and ring combination is studied using the membrane theory of shells to identify the membrane shear forces induced in the shell by the ring. These forces are then considered as loads applied to the intermediate ring stiffener. Vlasov's curved beam theory is used to derive closed form expressions for the variation of the stress resultants around the circumference to obtain a suitable strength design criterion for the stiffener.

3. STRESS AND DISPLACEMENT TRANSFER INTO INTERMEDIATE RING STIFFENERS

Topkaya and Rotter [3] determined the ideal location for an intermediate ring stiffener using the membrane theory of shells [5-7]. The loading on intermediate ring stiffeners can be obtained by solving for the reactions on the shell produced by a stiffener. All deformations, loading and stress resultants can be expressed in terms of a harmonic series around the circumference [6, 8] in order to solve the governing differential equations. In the case of discrete supports, the rapid decay in the effect of higher terms [9] means that the fundamental harmonic term of the column support force is sufficient to study the requirements of the ring stiffener, so the support force can be represented by

$$P_x = P_{xn} \cos n\theta \tag{2}$$

where P_x = external distributed axial line load applied to the base of the shell; P_{xn} = Fourier coefficient for the n^{th} harmonic of axial line load; and θ = circumferential coordinate.

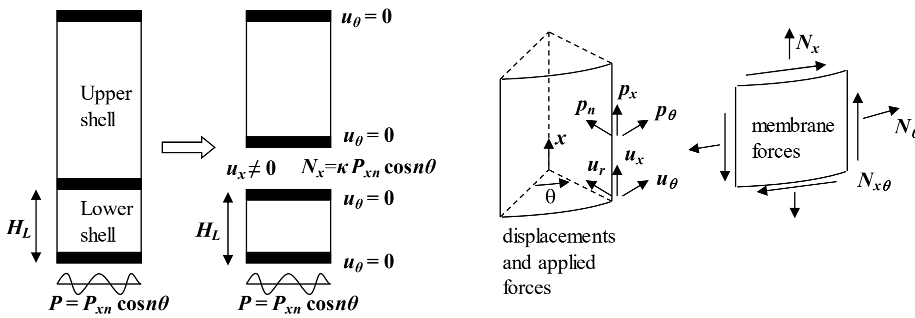

Fig. 2: Boundary conditions used in closed form solution, loading, displacements, and stress resultants in an element of the cylindrical shell

The cylindrical shell is here separated into two parts: an upper shell and a lower shell with the intermediate ring at their junction, as shown in *Fig. 2*, which also indicates the chosen boundary conditions. The lower shell, of height H_L, is subjected to the fundamental harmonic of the column support. The upper shell is assumed to be unloaded on its upper boundary and restrained against circumferential displacements by a ring. Topkaya and Rotter [3] demonstrated that the interface between the lower shell and upper shell will be free of both axial stress and axial displacements if an intermediate ring is placed at the ideal location. When the intermediate ring is placed below the ideal location some level of axial stress non-uniformity is present in the upper shell segment. In addition, the axial displacements no longer vanish, so the nonzero axial displacements can also be found at this interface.

Considering the cylindrical shell element shown in *Fig. 2*, the equilibrium equations are:

$$\frac{\partial N_x}{\partial x} + \frac{1}{r}\frac{\partial N_{x\theta}}{\partial \theta} + p_x = 0 \qquad \frac{\partial N_{x\theta}}{\partial x} + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta} + p_\theta = 0 \qquad N_\theta + r\,p_n = 0 \tag{3}$$

where N_x, N_θ, $N_{x\theta}$ = axial, circumferential and shear membrane stress resultants respectively; and p_x, p_θ, p_n = external distributed pressures in the axial, circumferential and radial directions respectively.

The strain, displacement and constitutive relationships can be written as:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = \frac{1}{Et}\left(N_x - \nu N_\theta\right) \qquad \varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r} = \frac{1}{Et}\left(N_\theta - \nu N_x\right) \tag{4}$$

$$\gamma_{x\theta} = \frac{1}{r}\frac{\partial u_x}{\partial \theta} + \frac{\partial u_\theta}{\partial x} = \frac{1}{Gt}N_{x\theta} = \frac{2(1+\nu)}{Et}N_{x\theta} \tag{5}$$

where u_x, u_θ, u_r = displacements in the axial, circumferential and radial directions respectively; ε_x, ε_θ = strains in the axial and circumferential directions respectively; $\gamma_{x\theta}$ = shear strain; ν = Poisson's ratio; E = modulus of elasticity; G = shear modulus; and t = thickness of the shell.

In the lower shell, eq. (3) may be solved sequentially by integrating in the x direction to obtain:

$$N_\theta = -r\,p_n \qquad N_{x\theta} = -\int\left(p_\theta + \frac{1}{r}\frac{\partial N_\theta}{\partial \theta}\right)dx + f_1(\theta) \qquad N_x = -\int\left(p_x + \frac{1}{r}\frac{\partial N_{x\theta}}{\partial \theta}\right)dx + f_2(\theta) \tag{6}$$

where $f_1(\theta)$, $f_2(\theta)$ = unknown functions of θ to be determined from two boundary conditions.

The general solution for the displacements of the shell may then be found as:

$$Etu_x = \int \left(N_x - \nu N_\theta\right) dx + f_3(\theta) \qquad Etu_r = Et\frac{\partial u_\theta}{\partial \theta} - r\left(N_\theta - \nu N_x\right) \tag{7}$$

$$Etu_\theta = 2\ (1+\nu)\int N_{x\theta} dx - \frac{Et}{r}\int \frac{\partial u_x}{\partial \theta} dx + f_4(\theta) \tag{8}$$

where $f_3(\theta)$, $f_4(\theta)$ = additional functions to satisfy the boundary conditions on the edges x = constant.

Where there is no surface loading on the shell $\left(p_x = p_\theta = p_n = 0\right)$, eq. (6) give:

$$N_\theta = 0 \qquad N_{x\theta} = f_1(\theta) \qquad N_x = -\frac{1}{r}\int\left(\frac{d}{d\theta} f_1(\theta)\right)dx + f_2(\theta) = -\frac{x}{r}\left(\frac{d}{d\theta} f_1(\theta)\right) + f_2(\theta) \tag{9}$$

At the base, x = 0, the axial membrane stress resultant is chosen as the fundamental harmonic of the discrete support, $N_x = -P_{xn}\cos n\theta$ (eq. (2)), leading to:

$$f_2(\theta) = -P_{xn}\cos n\theta \tag{10}$$

When the ring is placed at the ideal location, the axial stress vanishes at this height ($N_x = 0$), but when the ring is placed below the ideal location non-uniform axial stresses will still be present. As shown in *Fig. 2*, a certain proportion of the applied axial membrane stress resultant is assumed to be present at the interface. The ratio of the axial membrane stress resultant at the interface to the applied fundamental harmonic of the column support is here termed κ. Topkaya and Rotter [3] explored the magnitudes of axial membrane stress resultants that remain at this interface using many linear finite element analyses. The location of the intermediate ring, shell radius, number of supports and shell thickness ratios ($g = t_U/t_L$ where t_U and t_L are the thicknesses of the upper and lower shells respectively) were considered as the primary variables. *Fig. 3* shows the variation of the ratio of axial membrane stress resultants for the case of g = 0.5.

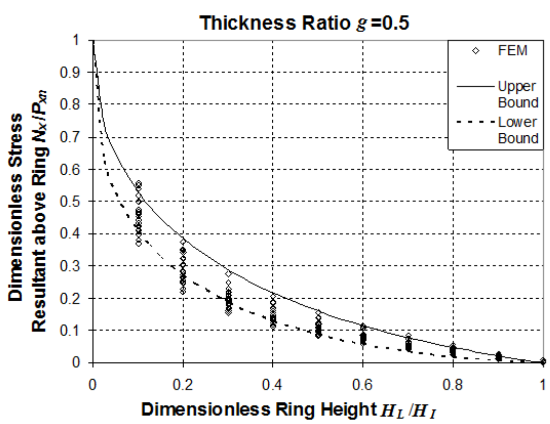

Fig. 3: Variation of axial stress resultant for various intermediate ring heights with upper to lower shell thickness ratio g =0.5

The following convenient lower and upper bound expressions can be developed to represent the data points:

$$\kappa = \frac{N_x}{P_{xn}} = 1 + g\left(\frac{H_L}{H_I}\right) - \left(1+g\right)\left(\frac{H_L}{H_I}\right)^m \tag{11}$$

with $m = 0.5g(2.34 - g)$ for upper bound $m = 0.4g(2.34 - g)$ for lower bound (12)

Considering an axial stress resultant $N_x = -\kappa P_{xn} \cos n\theta$ at the ring (x = H_L) leads to the following:

$$f_1(\theta) = -(1-\kappa)\frac{r}{H_L n} P_{xn} \sin n\theta \tag{13}$$

Inserting eqs (7) and (9) into eq. (8) yields the circumferential displacement as:

$$Etu_\theta = \frac{x^3}{6r^2}\left(\frac{d^2}{d\theta^2} f_1(\theta)\right) - \frac{x^2}{2r}\left(\frac{d}{d\theta} f_2(\theta)\right) + x\left[2(1+\nu)f_1(\theta) - \frac{1}{r}\left(\frac{d}{d\theta} f_3(\theta)\right)\right] + f_4(\theta) \tag{14}$$

At x = 0 and x = H_L, the boundary condition of zero circumferential displacements, $u_\theta = 0$, yields the two results:

$$f_4(\theta) = 0 \text{ and } f_3(\theta) = \left(\frac{(2+\kappa)H_L}{6} + \frac{2(1+\nu)(1-\kappa)r^2}{H_L n^2}\right) P_{xn} \cos n\theta \tag{15}$$

Inserting eqs (9) and (15) into eq. (7) yields the axial displacements as:

$$u_x = \frac{P_{xn} \cos n\theta}{Et}\left[\frac{(1-\kappa)x^2}{2H_L} - x + \frac{(2+\kappa)H_L}{6} + \frac{2(1+\nu)(1-\kappa)r^2}{H_L n^2}\right] \tag{16}$$

When the intermediate ring is placed at the ideal height the axial displacements and axial stress resultants at the interface vanish. The condition of κ = 0 with $u_x = 0$ at $x = H_L = H_I$ leads to the ideal location of the intermediate ring stiffener, previously expressed in eq. (1).

The reactions in the shell at this boundary can be treated as the loading exerted on the intermediate ring. Combining eqs (9) and (13) gives the following expression for the shear membrane stress resultant and the displacements at the interface (x = H_L) can be found from eq. (16) as:

$$N_{x\theta} = -(1-\kappa)\frac{r}{H_L n} P_{xn} \sin n\theta \quad u_x = \left[\frac{12(1+\nu)(1-\kappa)r^2 - H_L^2 n^2(1+2\kappa)}{6EtH_L n^2}\right] P_{xn} \cos n\theta \tag{17}$$

4. STRESS AND DISPLACEMENT TRANSFER INTO INTERMEDIATE RING STIFFENERS

4.1 Derivation of stress resultants – In plane behaviour

The six basic equilibrium equations for the curved beam element shown in *Fig. 4* can be expressed using the Vlasov's differential equations [10-11] as follows:

$$\frac{1}{r}\left[\frac{dQ_r}{d\theta} + Q_\theta\right] + q_r = 0 \qquad \frac{1}{r}\frac{dQ_x}{d\theta} + q_x = 0 \qquad \frac{1}{r}\left[\frac{dQ_\theta}{d\theta} - Q_r\right] + q_\theta = 0 \tag{18}$$

$$\frac{1}{r}\left[\frac{dM_r}{d\theta} + T_\theta\right] - Q_x + m_r = 0 \qquad \frac{1}{r}\frac{dM_x}{d\theta} + m_x + Q_r = 0 \qquad \frac{1}{r}\left[\frac{dT_\theta}{d\theta} - M_r\right] + m_\theta = 0 \tag{19}$$

where r = radius of the ring beam centroid; M_r = bending moment in the ring about a radial axis; M_x = bending moment in the ring about a transverse axis; T_θ = torsional moment in the ring; q_x, q_θ, q_r = distributed line loads per unit length in the transverse, circumferential and radial directions respectively; m_x, m_θ, m_r = distributed applied torques per unit circumference about the transverse, circumferential and radial directions respectively; Q_θ = circumferential tensile force in the ring; Q_x, Q_r = shear forces in the ring in transverse and radial directions respectively.

The six basic equilibrium equations can be reduced to three differential relationships for the case where the only loading is $q_\theta = N_{x\theta}$ (i.e. $q_r = q_x = m_r = m_\theta = m_x = 0$) as:

$$\frac{1}{r}\left[\frac{dT_\theta}{d\theta} - M_r\right] = 0 \qquad \frac{1}{r^2}\left[\frac{d^2M_r}{d\theta^2} + \frac{dT_\theta}{d\theta}\right] = 0 \qquad \frac{1}{r^2}\left[\frac{d^3M_x}{d\theta^3} + \frac{dM_x}{d\theta}\right] + q_\theta = 0 \tag{20}$$

Solution of eq. (20) and substituting obtained stress resultants into equilibrium equations reveal the following relationships for in-plane loading:

$$M_r(\theta) = 0 \qquad T_\theta(\theta) = 0 \qquad Q_x(\theta) = 0 \qquad Q_\theta(\theta) = -(1-\kappa)\frac{P_{xn}}{H_L}\frac{r^2}{(n^2-1)}\cos n\theta \tag{21}$$

$$M_x(\theta) = (1-\kappa)\frac{P_{xn}}{H_L}\frac{r^3}{n^2(n^2-1)}\cos n\theta \qquad Q_r(\theta) = (1-\kappa)\frac{P_{xn}}{H_L}\frac{r^2}{n(n^2-1)}\sin n\theta \tag{22}$$

These equations give expressions identical to those derived by Zeybek *et al*. [12] for the case where $H_L = H_I$ and $\kappa \to 0$.

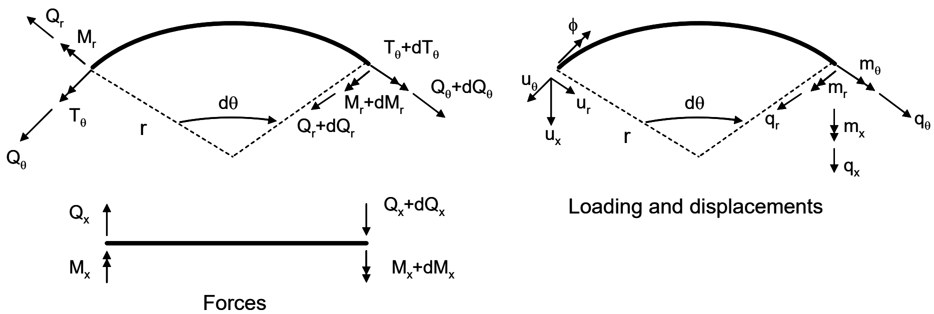

Fig. 4: Differential curved beam element and sign conventions

4.2 Derivation of stress resultants – Out-of-plane behaviour

The following expression define the force-deformation relationship [11]:

$$M_r = \frac{-EI_r}{r}\left(\frac{d^2u_x}{rd\theta^2} - \phi\right) \tag{23}$$

where u_x, u_θ, u_r = displacements in the vertical, circumferential, and radial directions respectively; ϕ = rotation; I_r, I_x= bending moment of inertia of the ring stiffener about radial and vertical axes respectively; I_w = warping constant of the ring stiffener; K_T = uniform torsional constant of the ring stiffener; A = cross sectional area of the ring stiffener; and G = shear modulus.

Inserting eq. (17) into eq. (23), substituting into equilibrium equations and considering ϕ = 0, and substituting obtained stress resultant into equilibrium equations reveal the following relationships:

$$M_r(\theta) = \frac{I_r\left[12(1+\nu)(1-\kappa)r^2 - H_L^2n^2(1+2\kappa)\right]}{6H_Lr^2t}P_{xn}\cos n\theta \tag{24}$$

$$T_\theta(\theta) = \frac{I_r\left[12(1+\nu)(1-\kappa)r^2 - H_L^2n^2(1+2\kappa)\right]}{6H_Lnr^2t}P_{xn}\sin n\theta \tag{25}$$

$$Q_x(\theta) = \frac{-I_r\left(n^2-1\right)\left[12(1+\nu)(1-\kappa)r^2 - H_L^2n^2(1+2\kappa)\right]}{6H_Lnr^3t}P_{xn}\sin n\theta \tag{26}$$

Eqs (24), (25) and (26) result in $M_r(\theta) = 0,\ T_\theta(\theta) = 0, Q_x(\theta) = 0$ for the case of $H_L = H_I$ and $\kappa \to 0$.

Considering typical ratios of the strong axis to weak axis elastic section moduli of rolled I sections, it is here recommended that the ratio of out-of-plane moment to in-plane moment should be limited to 10% ($M_r/M_x<0.1$). Neglecting the contribution of κ (i.e. $\kappa = 0$), the section can be selected by using the following expression for the second moment of area about the radial axis (I_r):

$$I_r < \frac{r^3 t}{20(1+\nu)n^2(n^2-1)\left(1-\left(\frac{H_L}{H_I}\right)^2\right)} \tag{27}$$

Having restricted the out-of-plane bending moment using eq. (27), the intermediate ring can be designed to resist out-of-plane moments of only 10% of the maximum in-plane bending moment.

5. COMPUTATIONAL VERIFICATION OF THE ABOVE EQUATIONS

The commercial finite element program, ANSYS v12.1 [13], was used to perform the numerical analysis. The computational time was reduced by modelling only a segment covering the angle π/n. Four-node shell elements (shell63) were employed to model the cylindrical shell. The intermediate ring stiffener was modelled using two-node beam elements (beam4). The modulus of elasticity was taken as 200 GPa and Poisson's ratio as 0.30. The cylinder base was subjected to loading in harmonic 4 (eq. (2)), corresponding to the number of equally spaced discrete supports. The loading at the bottom of the shell was chosen to give a maximum axial membrane stress of 100 MPa above the support. The silo structure analyzed here had a cylinder radius of 3000 mm and a height of 10000 mm. The lower and upper shell thicknesses were 6 mm and 3 mm respectively. A flexible intermediate ring where the out-of-plane second moment of area is defined by the upper limit given in eq. (27) was placed at half of the ideal height ($H_I/2$).

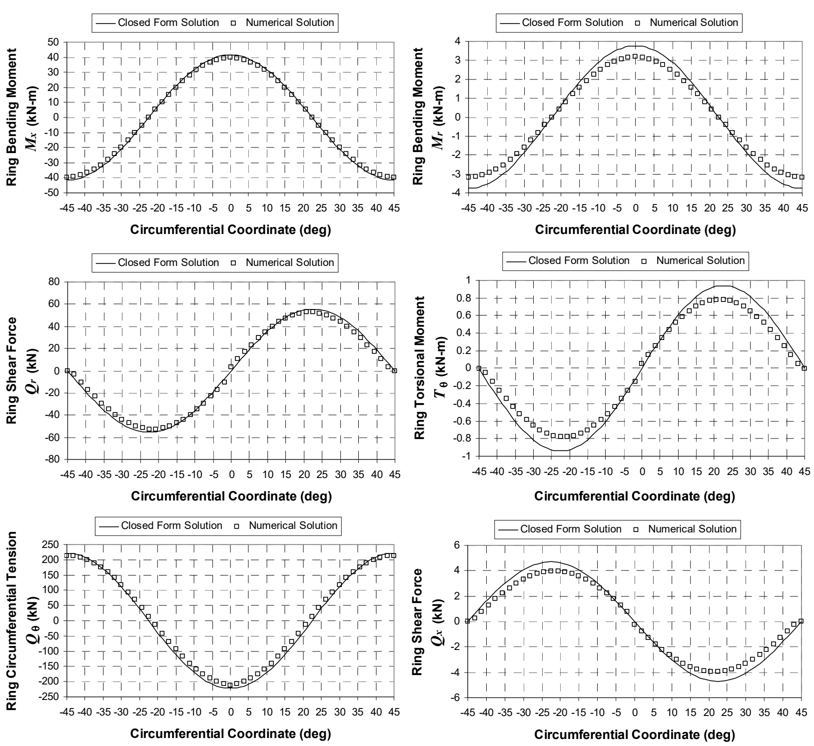

Fig. 5: Comparison of closed form solution with numerical solution for a flexible intermediate ring

The calculated variations of M_x, Q_r, Q_θ, M_r, T_θ and Q_x around the circumference, together with the predictions of the closed form solutions (eqs (21), (22), (24), (25) and (26)) for the intermediate ring are shown in *Fig. 5*. The lower bound expression (eq. (12)) was used in closed form solutions. The comparisons show that the above equations provide acceptably accurate solutions, with the largest differences being 4.8%, 5.1%, 4.8%, 18.2%, 20.2% and 18.4%, for maximum M_x, Q_r, Q_θ, M_r, T_θ and Q_x respectively

6. SUMMARY AND CONCLUSIONS

This paper has developed a criteria for the strength of intermediate ring stiffeners used in cylindrical silo shells resting on column-supported ring beams. The closed form expressions reveal that the intermediate ring stiffener is subjected to out-of-plane internal forces and bending moments in addition to the in-plane stress resultants when the ring is placed below the ideal height. The developed expressions were compared with numerical solutions and a good agreement has been demonstrated. The closed form solutions indicate that the out-of-plane internal forces and bending moments depend on the out-of-plane bending stiffness of the ring. For economical designs, it is proposed that the out-of-plane bending moment should be kept below 10% of the in-plane bending moment.

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