DYNAMIC ANALYSIS OF STEEL STRUCTURES USING THE MATERIAL POINT METHOD

Christos D. Sofianos
National Technical University of Athens
Institute of Structural Analysis & Aseismic Research
Zografou Campus, 15780 Athens, Greece
E-mail: sofianoschristos@yahoo.gr

Vlasis K. Koumousis
National Technical University of Athens
Institute of Structural Analysis & Aseismic Research
Zografou Campus, 15780 Athens, Greece
E-mail: vkoum@central.ntua.gr

1. ABSTRACT

In this work an explicit dynamic Material Point Method accounting for elastoplastic material behaviour is presented. Following the strain decomposition rule, the strain rates are uncoupled into an elastic and an inelastic part. The different stages in material behaviour (elastic loading or unloading, yielding and kinematic hardening) are tracked by two Heaviside type functions that act as switches. These functions incorporate the Von Mises yield criterion and result in a final form of the constitutive stress-strain relation that includes the tangent modulus of elasticity. An explicit scheme is employed advancing the solution from previously known states while satisfying the CFL stability criterion for the size of the time step. Numerical examples are presented that validate the proposed formulation and results are compared with existing dynamic Finite Element codes.

2. INTRODUCTION

The Material Point Method (MPM) [2] is an extension of the Particle in Cell (PIC) method. It is a hybrid method in a sense that it is based both on a Lagrangian and a Eulerian description. In Lagrangian methods the computational grid is embedded and deformed with the material. On the contrary in Eulerian methods the computational grid is fixed and the material moves through the grid. Eulerian methods are more appropriate in problems in which the material becomes heavily distorted.
The MPM tracks the deformation history of the material points and drastically reduces the numerical dissipation that can be found in Eulerian methods. In MPM since the grid is fixed and the particles track the deformation, problems related to mesh distortion and element entanglement are alleviated.

The method is considered a hybrid method since it takes advantage of both the Eulerian and Lagrangian description. At the beginning of each time step a Eulerian background grid is employed. The material is discretised into a number of material points that hold the properties and the state of the material (such as position, velocity, density, mass, stresses, strains etc.). The properties are then transferred to the background grid nodes where the governing equations are solved. The material points are then updated and the background grid is reset to its original form. Although the background grid nodes can be moved it is not necessary and is often avoided. This happens mainly because in a structured grid the identification of the element that each material point lies in is straightforward and computationally inexpensive. This is contrast with mesh-free methods like the Smoothed Particle Hydrodynamics where the nearest neighbour search takes a significant percentage of the computational time.

3. MPM ALGORITHM

A continuum body Ω is discretized by a set of N_p material points in an Eulerian grid composed of *N* quadrilateral 4 node elements for the background grid (Figure 1).

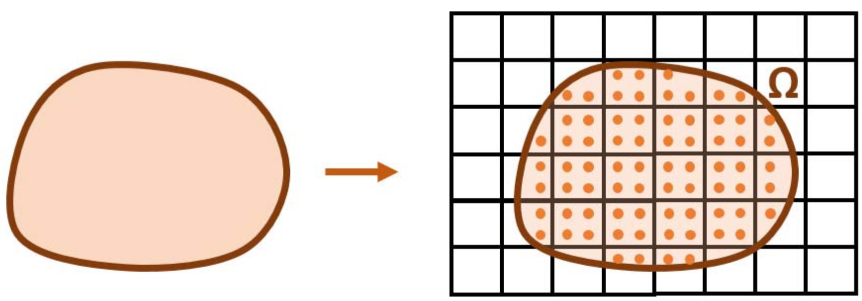

Fig. 1: MPM discretization

The material points hold the properties of the body: position, velocity, mass, stress, strain etc. The governing equations that need to be solved are the conservation equations, the constitutive equation, kinematic and boundary conditions and initial values. These equations are the following:

$$
\begin{aligned}
&\text{Mass conservation:} && pJ = p_0 \\
&\text{Momentum conservation:} && \sigma \cdot \nabla + pb = p\dot{v} \\
&\text{Energy conservation:} && p\dot{e} = D : \sigma + ps + \nabla \cdot \left(k \nabla T\right) \\
&\text{Constitutive equation:} && \sigma = \sigma\left(D, \sigma, \text{etc}\right) \\
&\text{Rate of deformation:} && D = \frac{1}{2}\left(L + L^T\right)
\end{aligned}
\qquad (1)
$$

Where *p* is the current density, *b* is the body force per unit mass acting on the body and σ is the Cauchy stress. In the following M_p denotes the mass of the particle, *M* denotes the mass at the node, x_p is the particle position, *E* the Young's modulus and p_p the particle mass density.

The main algorithm of the MPM is the following:

1.Background grid is reset and all its variables are set to zero

2.The element number that each material point lies in is identified.

3.The shape functions and their derivatives are calculated. For each material point 16 shape function values are calculated and 32 derivatives since every material point maps its properties to 16 nodes using the cubic B-Splines shape functions. The cubic B-Splines shape functions have been shown to reduce quadrature errors as well as eliminate the grid crossing errors due to their shape function gradients being continuous [3], [4].

4.The material point masses, momenta and internal forces are mapped to the nodes of the background grid

$$m_i = \sum_{p=1}^{N_p} M_p N_i$$
$$(mv)_i = \sum_{p=1}^{N_p} (Mv)_p N_i \quad (2)$$
$$F_i^{\text{int}} = -\sum_{p=1}^{N_p} \frac{M_p}{p_p} \sigma_p \nabla N_i$$

5.The external forces are also mapped to the grid nodes. In the case where the external forces are applied directly into grid nodes no mapping is necessary and their values are added in the corresponding degrees of freedom.

6.The total grid force vector is calculated:

$$F_i = F_i^{ext} + F_i^{\text{int}} \quad (3)$$

7.The grid momenta are updated using an explicit formulation:

$$(mv)_i^t = (mv)_i^{t-1} + F_i dt \quad (4)$$

8.Essential boundary conditions are applied to the grid nodes

9.From the updated momenta at the grid nodes the acceleration and velocities of particles are updated:

$$v(x_p)^t = v(x_p)^{t-1} + \left(\sum_{i=1}^{N} \frac{F_i N_i}{m_i} \right) dt$$
$$x_p^{\ t} = x_p^{\ t-1} + \left(\sum_{i=1}^{N} \frac{(mv)_i N_i}{m_i} \right) dt \quad (5)$$

10. Employing the Modified Update Stress Last (MUSL) scheme the final step is to recalculate the grid nodal momentum based on the new particle velocities and from there calculate the particle strain and stress increments as:

$$
\begin{aligned}
&(mv)_i = \sum_{p=1}^{N_p} M_p v(x_p) N_i \\
&v_i = \frac{(mv)_i}{m_i} \\
&\Delta\varepsilon_p = \sum_{i=1}^{N} v(x_p) \nabla N_i dt \\
&\Delta\sigma_p = E\Delta\varepsilon_p
\end{aligned}
\tag{6}
$$

3. PLASTICITY MODEL

According to the theory of classical plasticity the governing equations are the flow rule, the yield condition, the consistency condition and the hardening rule. For the yield condition a Von Mises criterion is adopted. The flow rule expresses the plastic strain tensor $\{\dot{\varepsilon}^p\}$ in terms of the plastic multiplier $\dot{\lambda}$ as:

$$
\{\dot{\varepsilon}^p\} = \dot{\lambda} \frac{\partial\Phi(\{\sigma\})}{\partial\{\sigma\}} \tag{7}
$$

The relation that connects stresses and strains in rate form is the following:

$$
\{\dot{\sigma}\} = [D]\{\dot{\varepsilon}^{el}\} \tag{8}
$$

According to the additive decomposition of the strain rates and substituting the relation of the plastic strain rates we can derive the following relation:

$$
\{\dot{\sigma}\} = [D]\left\{\{\dot{\varepsilon}\} - \dot{\lambda}\frac{\partial\Phi(\{\sigma\})}{\partial\{\sigma\}}\right\} \tag{9}
$$

During yielding $\dot{\Phi} = 0$ and $\dot{\lambda} \geq 0$ and considering the consistency condition:

$$
\left(\frac{\partial\Phi(\{\sigma\})}{\partial\{\sigma\}}\right)^T \{\dot{\sigma}\} + \left(\frac{\partial\Phi(\{\sigma\})}{\partial\{\sigma\}}\right)^T \{\dot{\eta}\} = 0 \tag{10}
$$

The hardening law is defined as a relation between the back-stress tensor and the plastic strain tensor. In any case it is derived as a function of the plastic multiplier:

$$
\{\dot{\eta}\} = \dot{\lambda} G(\{\eta\}, \Phi) \tag{11}
$$

Using the previous relations and solving for the plastic multiplier we can derive the final expression for the plastic multiplier:

$$\dot{\lambda}=\left(-\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}G\left(\{\eta\},\Phi\right)+\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}[D]\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{-1}\cdots$$
$$\cdots\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}[D]\{\dot{\varepsilon}\} \quad (12)$$

The previous relation holds when yielding has occurred. In order to generalize the plastic multiplier in the whole domain and smooth the transition from the elastic to the inelastic regime we introduce the following Heaviside type functions [5], [1]:

$$H_1=\left|\frac{\Phi}{\Phi_0}\right|^{n} \quad ,H_2=0.5+0.5sign\left(\{\varepsilon\}^{T}\{\dot{\sigma}\}\right) \quad (13)$$

The first function H_1 controls the smoothness of the curve from the elastic to the inelastic regime while the second function H_2 controls the loading or unloading/reloading branches. These Heaviside type functions are introduced directly inside the relation of the plastic multiplier effectively extending its domain both for the elastic as well as the inelastic region.

$$\dot{\lambda}=H_1H_2\left(-\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}G\left(\{\eta\},\Phi\right)+\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}[D]\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{-1}\cdots$$
$$\cdots\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}[D]\{\dot{\varepsilon}\} \quad (14)$$

Finally, by substituting relation (14) of the plastic multiplier directly into equation (9) and after some algebra we can derive the final expression for the tangent elasticity matrix as:

$$\{\dot{\sigma}\}=[E_t]\{\dot{\varepsilon}\}$$
$$[E_t]=[D]\left([I]-\left|\frac{\Phi}{\Phi_0}\right|^{n}\left(0.5+0.5sign\left(\{\varepsilon\}^{T}\{\dot{\sigma}\}\right)\right)[R]\right) \quad (15)$$

where $[R]$ is the interaction matrix:

$$[R]=\left(-\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}G\left(\{\eta\},\Phi\right)+\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}[D]\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{-1}\cdots$$
$$\cdots\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)\left(\frac{\partial\Phi\left(\{\sigma\}\right)}{\partial\{\sigma\}}\right)^{T}[D] \quad (16)$$

Now, for the MPM the classic elasticity matrix in (6) is replaced by the tangent matrix from equation (15) effectively incorporating plasticity into the MPM framework without the need for a demanding bookkeeping mechanism.

4. NUMERICAL EXAMPLES

To verify the proposed formulation a steel frame structure is considered [6]. Material is steel with Young's modulus of 207 GPa, yield strength of 470 MPa and 5% hardening is considered. The frame is fixed and the load applied is constant over time with a value of 444.82 kN. The columns are W12x120 while the beam is W16x36. In Figure 2 the displacement time history of the left node of the frame is plotted both when the frame is considered elastic (left) and when plasticity is taken into account (right). Results are compared to inelastic beam finite element code and are in good agreement [6]. The slight differences that are observed for the inelastic case can be attributed to the difference in the parameter value *n* from equation (15) that controls the smoothness from the elastic to the inelastic region. With the black solid line and dots are the results using beam elements while the MPM results are plotted in red solid lines. Finally, in Figure 3 the deformed shape of the frame is presented.

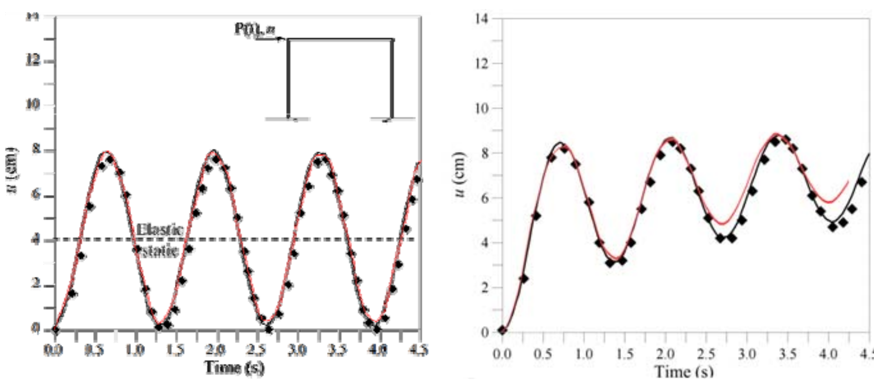

Fig. 2: Time history of the displacement of the upper left node of the frame (elastic and elastoplastic behavior) [6].

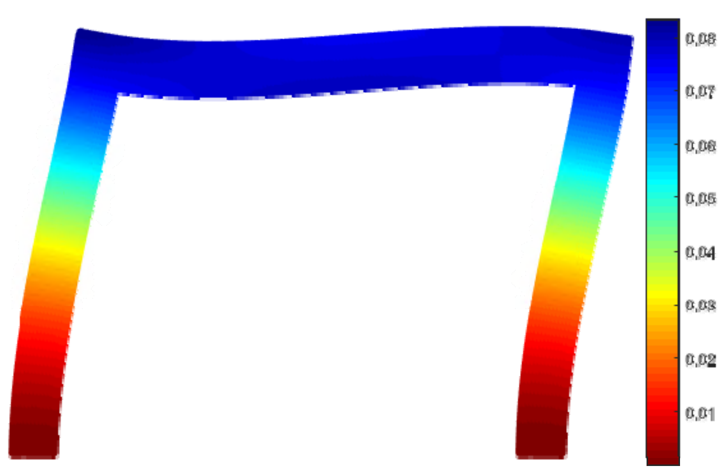

Fig. 3: Deformed shape for the frame at maximum amplitude.

5. CONCLUSIONS

The Material Point Method has been extended to account for hysteresis and plasticity for the explicit dynamic analysis of structures. The plasticity model is based on the extension of the plastic multiplier in the whole domain to account for smooth transition from the elastic to the inelastic region. The whole formulation results in the tangent elasticity matrix for each material point that controls its behaviour and is efficiently incorporated into the Material Point Method. Results that have been presented that verify the proposed formulation.

6. REFERENCES

[1] Bouc R. "Forced vibration of mechanical system with hysteresis". *Proc. 4th Conf. on Nonlinear Oscillation*, Prague

[2] Sulsky D, Chen Z and Schreyer HL "A particle method for history-dependent materials" *Comput. Methods, Appl. Mech. Eng*. 118,179 (1994).

[3] Andersen S, Andersen L "Analysis of spatial interpolation in the material-point method" *Computers and Structures* 88, 506-518 (2010).

[4] Steffen M, Kirby RM, Berzins M. "Analysis and reduction of quadrature errors in the material point method (MPM)" *International Journal for Numerical Methods in Engineering*;76:922–48 (2008).

[5] Triantafyllou S, Koumousis V, "Bouc-Wen Hysteretic Plane Stress Element" *Journal of Engineering Mechanics*, 138, 235-246 (2012).

[6] Marur SR, Kant T "A stress correction procedure for the analysis of inelastic frames under transient dynamic loads" *Computers & Structures*, 50 (5), pp. 603-613

ΔΥΝΑΜΙΚΗ ΑΝΑΛΥΣΗ ΜΕΤΑΛΛΙΚΩΝ ΚΑΤΑΣΚΕΥΩΝ ΜΕ ΤΗ ΜΕΘΟΔΟ ΥΛΙΚΟΥ ΣΗΜΕΙΟΥ

Χρήστος Δ. Σοφιανός
Εθνικό Μετσόβιο Πολυτεχνείο
Εργαστήριο Στατικής & Αντισεισμικών Ερευνών
Ζωγράφου, 15780 Αθήνα, Ελλάδα
e-mail: sofianoschristos@yahoo.gr

Βλάσης Κ. Κουμούσης
Εθνικό Μετσόβιο Πολυτεχνείο
Εργαστήριο Στατικής & Αντισεισμικών Ερευνών
Ζωγράφου Campus, 15780 Αθήνα, Ελλάδα
e-mail: vkoum@central.ntua.gr

ΠΕΡΙΛΗΨΗ

Η Μέθοδος Υλικού Σημείου (Material Point Method - MPM) προέρχεται από τις μεθόδους σωματιδίου σε κελί (Particle In Cell - PIC). Η κατασκευή διακριτοποιείται σε ένα σύνολο υλικών σημείων που διατηρούν τις ιδιότητες και την κατάσταση του υλικού όπως τάσεις, παραμορφώσεις, θέσεις, ταχύτητες κτλ. Στη συνέχεια, οι ιδιότητες αυτές παρεμβάλλονται με τη χρήση συναρτήσεων σχήματος σε ένα πλέγμα στο οποίο και επιλύονται οι εξισώσεις που διέπουν το πρόβλημα. Για προβλήματα δυναμική οι εξισώσεις αυτές είναι η διατήρηση της ενέργειας, η διατήρηση της μάζας, η διατήρηση της ορμής καθώς και οι καταστατικές εξισώσεις και οι συνοριακές και αρχικές συνθήκες. Μετά την επίλυση στου κόμβους του πλέγματος ακολουθεί η παρεμβολή των νέων ιδιοτήτων πίσω στα υλικά σημεία τα οποία μετακινούνται. Το πλέγμα χρησιμοποιείται μόνο για την επίλυση των εξισώσεων και στο τέλος κάθε χρονικού βήματος επαναφέρεται στην αρχική του μορφή. Αυτό αποτελεί και ένα πλεονέκτημα της μεθόδου αφού δεν δημιουργούνται προβλήματα υπερβολικής παραμόρφωσης των στοιχείων καθώς και αποφεύγεται η χρονοβόρα εύρεση γειτόνων που χρησιμοποιείται σε άλλες διακριτές μεθόδους.

Σε αυτή την εργασία η Μέθοδος Υλικού Σημείου επεκτείνεται για τη μελέτη ελαστοπλαστικών συμπεριφορών που είναι κυρίαρχες στις μεταλλικές κατασκευές. Η καταστατική σχέση τάσεων – παραμορφώσεων προκύπτει από το διαχωρισμό των παραμορφώσεων στο ελαστικό και το πλαστικό μέρος τους, που ισχύει και για την περίπτωση των παραγώγων τους. Ο πλαστικός πολλαπλασιαστής επεκτείνεται με τη χρήση δύο συναρτήσεων τύπου Heaviside σε όλο το πεδίο για την ολοκληρωμένη καταγραφή της ελαστικής συμπεριφοράς, της διαρροής καθώς και της αποφόρτισης και επαναφόρτισης. Το κριτήριο διαρροής είναι Von Mises. Όλα τα παραπάνω ενσωματώνονται στον υπολογισμό του εφαπτομενικού μέτρου ελαστικότητας για κάθε υλικό σημείο.